GAUSS-MANIN DETERMINANTS FOR RANK 1 IRREGULAR CONNECTIONS ON CURVES

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Abstract. Let $f : U \to \mathrm{Spec}\,(K)$ be a smooth open curve over a field $K \supset k$, where k is an algebraically closed field of characteristic 0. Let $\nabla : L \to L \otimes \Omega^1_{U/k}$ be a (possibly irregular) absolutely integrable connection on a line bundle L. A formula is given for the determinant of de Rham cohomology with its Gauß-Manin connection $\Big(\det Rf_*(L\otimes\Omega^1_{U/K}),\ \det \nabla_{GM}\Big)$. The formula is expressed as a norm from the curve of a cocycle with values in a complex defining algebraic differential characters [7], and this cocycle is shown to exist for connections of arbitrary rank.

Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

Bertrand Russell

1. Introduction

Let $f : U \to \mathrm{Spec}\,(K)$ be a smooth open curve over a field $K \supset k$, where k is an algebraically closed field of characteristic zero. Let $\nabla : L \to L \otimes \Omega^1_{U/k}$ be a possibly irregular absolutely integrable (or vertical, see definition 2.16) connection on a line bundle L. The Riemann-Roch problem in this context is to describe characteristic classes for the relative de Rham cohomology $Rf_*(L \otimes \Omega^*_{U/K})$ as a (virtual) vector space over K with an integrable connection, in terms of data on U. The 0-th characteristic class, the Euler characteristic $\dim R^0 - \dim R^1$, is well-known to be given by

$$2 - 2g - n - \sum_i \max(0, m_i - 1) \tag{1.1}$$

where g is the genus of the complete curve C, n is the number of missing points, and m_i is the order of the polar part of the connection at the i-th missing point. The purpose of this article is to give a formula for the

Date: August 19, 1999, December 9, 2000.
1991 *Mathematics Subject Classification.* Primary 14C40 19E20 14C99.

first characteristic class, which is the determinant of the Gauß-Manin connection on the relative de Rham cohomology of the line bundle,

$$\Big(\det(Rf_*(L\otimes\Omega^*_{U/K})), \det\nabla_{GM}\Big). \tag{1.2}$$

When $U = C$, so the connection has no poles, the formula given in [2] is

$$\Big(\det(Rf_*(L\otimes\Omega^*_{C/K})), \nabla_{GM}\Big) = -f_*((L,\nabla)\cdot c_1(\Omega^1_{C/K})). \tag{1.3}$$

Concretely, if one has $c_i \in C(K)$ with $\sum c_i$ a 0-cycle in the linear series representing $\Omega^1_{C/K}$, then the determinant is given by restricting L with its connection to each c_i and then tensoring the resulting lines with connection together.

When the connection ∇ has at worst regular singular points at the points in $D := C - U$ there is an analogous formula using linear series given by divisors of rational sections s of $\Omega^1_{C/K}(D)$ satisfying the rigidity condition $\mathrm{res}_D(s) = 1$. Indeed, these formulas are valid also for higher rank connections. One takes the determinant at zeroes and poles of s.

In the case of irregular singular points, a similar formula is possible, but the rigidification taken must depend on the polar part of the connection. Let $(\mathcal{L},\nabla)$ be an extension of (L,∇) to C, $\mathcal{D} = \sum_i m_i D_i$ be a divisor with multiplicities $m_i \geq 1$ supported in $C - U$ such that the relative connection

$$\nabla_{/K} : \mathcal{L} \to \mathcal{L}\otimes\Omega^1_{C/K}(\mathcal{D}) \tag{1.4}$$

yields a complex quasiisomorphic to $j_*L \to j_*L\otimes\Omega^1_{(C-D)/K}$ and has poles at all points D_i. Then $\nabla_{/K}$ does not factor through

$$\nabla_{/K} : \mathcal{L} \to \mathcal{L}\otimes\Omega^1_{C/K}(\mathcal{D}-D_i) \tag{1.5}$$

for any i. Writing $\mathcal{D}$ also for the artinian subscheme of C determined by $\mathcal{D}$, this implies that $\nabla_{/K}$ induces a *function linear isomorphism*

$$\nabla|_{\mathcal{D}} : \mathcal{L}|_{\mathcal{D}} \xrightarrow{\cong} \mathcal{L}\otimes\Omega^1_{C/K}(\mathcal{D})|_{\mathcal{D}} \tag{1.6}$$

Because these maps are function linear, we may cancel the lines $\mathcal{L}|_{\mathcal{D}}$ and deduce canonical elements $\mathrm{triv}_\nabla \in \Omega^1_{C/K}(\mathcal{D})|_{\mathcal{D}}$. We view triv_∇ as a trivialization of $\Omega^1_{C/K}(\mathcal{D})$ along $\mathcal{D}$. It is known ([6], Appendix B) that the coboundary of triv_∇ in $H^1(C,\omega_{C/K}) \cong K$ is given by the degree of $\mathcal{L}$. Our main result is:

Theorem 1.1. *Let notation be as above. Assume all D_i are defined over K and some $m_i \geq 2$. Because we are only concerned with the cohomology over $X - \mathcal{D}$, we can take $\mathcal{L}$ of degree 0 so that the element* triv_∇ *can be lifted to $H^0(C,\Omega^1_{C/K}(\mathcal{D}))$. Let s be any such lifting, and*

write (s) for the divisor of s as a section of $\Omega^1_{C/K}(\mathcal{D})$. Note the support of (s) is disjoint from $\mathcal{D}$. Then

$$(1.7)\quad \det(Rf_*(L\otimes\Omega^*_{U/K})), \det(\nabla_{GM}) \cong \\ - f_*(L\cdot(s)) + \tau(L) \in \Omega^1_K/d\log K^*.$$

Here $\tau(L)$ is a 2-torsion term which can be written

$$\tau(L) = \sum_i \frac{m_i}{2} d\log(g_{i,0}) \in \frac{1}{2} d\log(K^\times)/d\log(K^\times)$$

where the connection $\nabla_{/K} = (g_{i,0} + g_{i,1}z_i + \dots)dz_i/z_i^{m_i}$ for a local coordinate z_i at $D_i \in \mathcal{D}$.

Note that $\Omega^1_K/d\log K^\times$ is the group of isomorphism classes of rank 1 connections on Spec (K). Our assumption that points of $\mathcal{D}$ are defined over K is made to avoid complications involving generalized jacobians in §2. We remark, of course, that part of our task will be to give a precise definition of the right hand side of the formula of the theorem. It will appear as a product followed by a trace, and this definition does not depend on the particular choice of $\mathcal{L}$ above. In particular, this gives a formulation if we don't assume that $\mathcal{L}$ is of degree 0, and also if we don't assume that $m_i \geq 2$ for at least one i, that is if ∇ has regular singular points (see theorem 4.6). The precise general formulation of our theorem is in 4.8. In the case that (s) is a sum of K-points c_i, one may simply take the tensor product of the lines with connection $L|_{c_i}$. The right hand side of the formula depends only on the equivalence class of (s) in a generalized Picard (or divisor class) group of line bundles with trivializations along $\mathcal{D}$. Thus, by analogy with (1.3), it is natural to write formula (1.7) in the form

$$(1.8)\quad \Big((\det(Rf_*(L\otimes\Omega^*_{U/K})), \det(\nabla_{GM})\Big) \cong \\ f_*\Big(L\cdot c_1(\Omega^1_{C/K}(\mathcal{D}), \mathrm{triv}_\nabla))\Big)^{-1} + \tau(L).$$

The classical Riemann-Roch pattern begins to break down in that the characteristic class $c_1(\Omega^1_{C/K}(\mathcal{D}), \mathrm{triv}_\nabla)$ depends on more than just the geometry of $f: U \to \mathrm{Spec}\,(K)$. This reflects the fact that the de Rham cohomology of an irregular connection depends on more than topology.

There is an analogy here with the case of ℓ-adic sheaves. If $\mathcal{E}$ is an unramified ℓ-adic sheaf on a complete curve C over a finite field $\mathbb{F}_q$, then the global epsilon factor is given by

$$(-F_q|\det(H^*_{\text{ét}}(C_{\overline{\mathbb{F}}}, \mathcal{E}))) = \det(\mathcal{E})^{-1}\cdot c_1(\Omega^1_{C/\mathbb{F}_q}).$$

The basic result in the ramified ℓ-adic case ([8]) is that the global epsilon factor can be written as a product of local terms corresponding to points on the curve where the sheaf ramifies or where a chosen meromorphic 1-form has zeroes or poles. We suspect formula (1.8) is analogous to a classical formula for Gauß sums

$$g(c,\psi) = \sum_{a \in (\mathcal{O}/\mathfrak{f})^\times} c(a)\psi(a)$$

where $\mathfrak{f} \subset \mathcal{O}$ is an ideal in the ring of integers in a local field, c (resp. ψ) is a character of $(\mathcal{O}/\mathfrak{f})^\times$ (resp. $(\mathcal{O}/\mathfrak{f})^+$), and both c and ψ have conductor $\mathfrak{f}$. If the residue field of $\mathcal{O}$ has q elements with q odd, one finds

$$g(\epsilon,\psi) = \begin{cases} q^n c(x) & \mathfrak{f} = \mathfrak{m}^{2n},\ n \geq 1 \\ q^n c(x)\sigma & \mathfrak{f} = \mathfrak{m}^{2n+1},\ n \geq 1. \end{cases}$$

In this formula $x \in \mathcal{O}/\mathfrak{f}$ is a suitable point, $\sigma = \zeta\sigma_0$ with $\zeta^q = 1$ and $\sigma_0^2 = \left(\frac{-1}{\mathbb{F}_q}\right)$. (Here σ_0 is a quadratic Gauß sum.)

Our proof follows the main idea of Deligne [3]. For computing the ϵ-factor associated to a rank one Galois representation on a curve, he expresses the determinant of the cohomology as the cohomology on a symmetric product of $(C - D)$ and reduces the computation to the geometry of the generalized jacobian. In the geometric situation one is further able to express the determinant Gauß-Manin connection as the connection arising by restricting a certain translation-invariant connection to one specific K-point of the generalized jacobian. The essential point seems to be that the de Rham cohomology of a connection of the form $d + \omega$ on a trivial bundle is somehow concentrated at the points where $\omega = 0$.

In section 4 we reinterpret the Riemann-Roch formula in terms of a pairing (4)

$$\begin{aligned} \cup : \mathbb{H}^1(C, \mathcal{O}_C^* \to \mathcal{O}_{\mathcal{D}}^*) \times \mathbb{H}^1(C, \mathcal{O}_C^* \to \Omega_C^1\langle D\rangle(\mathcal{D}')) \\ \to \mathbb{H}^2(C, \mathcal{K}_2 \to \Omega_C^2) \end{aligned} \tag{1.9}$$

and a trace map (4.7)

$$\mathrm{Tr}\ : \mathbb{H}^2(C, \mathcal{K}_2 \to \Omega_C^2) \to \Omega_K^1/d\log K^*.$$

In section 5 we give an analogous "non-commutative" product formula in the higher rank case which we conjecture calculates the determinant connection in the generic situation when the connection defines local isomorphisms $E|_{\mathcal{D}} \cong E|_{\mathcal{D}} \otimes \omega_{\mathcal{D}/K}$ (see (5.3)) and the poles of the absolute connection behave well (see (5.1)). We verify the formula has the appropriate invariance properties. We also show that there is a more

general higher rank product of which it is a special case. Finally, in section 6 we give a general formula which calculates the group of isomorphism classes of irregular, integrable, rank 1 connections in higher dimensions on a smooth projective variety.

We apologize for not expressing our results in the modern language of $\mathcal{D}$-modules, but in fact for the study of Gauß-Manin determinants there is little gain in passing from connections to $\mathcal{D}$-modules. Also, rigidity for connections means that the Gauß-Manin determinant connection is determined by its value at the generic point on the base, so we may work with curves over a function field $\mathrm{Spec}\,(K)$.

It is our pleasure to acknowledge the intellectual debt we owe in this work to P. Deligne. We are also grateful to the Humboldt foundation for financing which enabled us to work together.

2. Connections and forms on Generalized Jacobians

Throughout this paper C will be a smooth projective curve over a field K containing an algebraically closed subfield k of characteristic 0, and $\mathcal{D} = \sum m_i c_i$ is a divisor on C, with $c_i \in C(K)$. We write $G = J_{\mathcal{D}}$ for the generalized Jacobian parametrizing isomorphism classes of degree 0 line bundles on C with trivialization along $\mathcal{D}$. Fixing a K-rational point

$$c_0 \in (C - D)(K),$$

there is a cycle map $i : C - D \to J_{\mathcal{D}}$ associating to a closed point $x \in C$ with $[K(x) : K] = n$ the class of the line bundle $\mathcal{O}(x - nc_0)$ together with the trivialization $b|_{\mathcal{D}} \circ (a|_{\mathcal{D}})^{-1}$, where

$$\mathcal{O}_C \overset{a}{\hookleftarrow} \mathcal{O}_C(-nc_0) \overset{b}{\hookrightarrow} \mathcal{O}_C(x - nc_0)$$

are the natural maps.

The aim of this section is to describe invariant line bundles with connection on $J_{\mathcal{D}}$, comparing them via the cycle map i to line bundles with connection on $(C - D)$ with a certain irregularity behavior along D.

When the line bundle in question is the trivial bundle, this amounts to studying invariant (absolute) differential forms on the generalized jacobian, so we should start with that. Before doing so, however, it is necessary to understand global functions on the generalized jacobian. We write

$$G \twoheadrightarrow G_0 \twoheadrightarrow J \tag{2.1}$$

where J is the usual Jacobian of C, and G_0 is a semi-abelian variety. We have extensions

$$0 \to T \to G_0 \to J \to 0 \tag{2.2}$$

$$0 \to \mathbb{V} \to G \to G_0 \to 0 \tag{2.3}$$

Here $\mathbb{V}$ is a vector group (isomorphic to $\mathrm{Spec}\,(\mathrm{Sym}(V^*))$ for some vector space V) and T is a torus, i.e. $T_{\bar{K}} \cong \mathbb{G}_m^r$.

Lemma 2.1. *The semi-abelian variety G_0 admits a universal vectorial extension*

$$0 \to \mathbb{W} \to \mathcal{G} \to G_0 \to 0. \tag{2.4}$$

In fact, this extension is given by the pullback to G_0 of the universal vectorial extension over J. In particular, $\mathbb{W} = \Gamma(J^\vee, \Omega^1_{J^\vee/K}) \otimes \mathbb{G}_a$.

Proof. It will suffice to show the pullback vectorial extension is universal. Since $\mathrm{Ext}^1(T, \mathbb{G}_a) = (0) = \mathrm{Hom}(T, \mathbb{G}_a)$, any extension of G_0 by a vector group $\mathbb{W}$ is pulled back from a unique extension of J by $\mathbb{W}$. This extension of J is a pushout from the universal vectorial extension, so the same holds for the pullbacks to G_0. $\square$

Lemma 2.2. *Let $\pi : G_0 \to J$ be an extension of J by T as above. There exists, possibly after a finite field extension, a quotient torus $T \twoheadrightarrow S$ and a diagram*

$$\begin{array}{ccc} T & \hookrightarrow & G_0 \\ \mathit{surj.} \downarrow & \swarrow a & \\ S & & \end{array} \tag{2.5}$$

such that

$$H^i(G_0, \mathcal{O}_{G_0}) \cong H^i(J, \mathcal{O}_J) \otimes_K H^0(S, \mathcal{O}_S) \tag{2.6}$$

Proof. There is a boundary map

$$\partial : \mathrm{Hom}_{\bar{K}}(T, \mathbb{G}_m) \to \mathrm{Ext}^1_{\bar{K}}(J, \mathbb{G}_m) \tag{2.7}$$

Define $N := \ker(\partial) \subset M := Hom_{\bar{K}}(T, \mathbb{G}_m)$. Let $S = \mathrm{Hom}(N, \mathbb{G}_m)$ be the torus with character group N. For $m \in M$ let $L(m)$ be the line bundle on $J_{\bar{K}}$ corresponding under the map (2.7). As an $\mathcal{O}_{J_{\bar{K}}}$-algebra

$$\pi_* \mathcal{O}_{G_{0,\bar{K}}} \cong \oplus_{m \in M} L(m) \cong H^0(\mathcal{O}_S) \otimes \Big(\oplus_{m \in M/N} L(m) \Big) \tag{2.8}$$

The map a in the diagram (2.5) comes from the above inclusion

$$H^0(\mathcal{O}_S) \otimes L(0) \subset \pi_* \mathcal{O}_{G_0}.$$

For $m \in M/N$, (as is well known, cf. [9] III 16), $L(m)$ has trivial cohomology in all degrees unless $m = 0$. The proposition follows by taking cohomology of (2.8). $\square$

Lemma 2.3. *Let notation be as above. Let*

$$0 \longrightarrow H^0(J^\vee, \Omega^1_{J^\vee/K}) \otimes \mathbb{G}_a \longrightarrow \mathcal{G} \xrightarrow{p} G_0 \longrightarrow 0 \tag{2.9}$$

be the universal vectorial extension. Then

$$H^0(\mathcal{G}, \mathcal{O}_\mathcal{G}) \cong H^0(G_0, \mathcal{O}_{G_0}) \tag{2.10}$$

Proof. The $\mathcal{O}_{G_0}$-algebra $p_*\mathcal{O}_\mathcal{G}$ is filtered, with

$$gr_i p_*\mathcal{O}_\mathcal{G} = fil_i/fil_{i-1} \cong \mathrm{Sym}^i(H^0(J^\vee, \Omega^1_{J^\vee/K})^*) \otimes \mathcal{O}_{G_0} \tag{2.11}$$

With respect to the exact sequences

$$0 \longrightarrow fil_{i-1} \longrightarrow fil_i \longrightarrow gr_i \longrightarrow 0 \tag{2.12}$$

it suffices to show the boundary map

$$b : \mathrm{Sym}^i(H^0(J^\vee, \Omega^1_{J^\vee/K})^*) \otimes H^0(G_0, \mathcal{O}_{G_0}) \to H^1(G_0, fil_{i-1}) \tag{2.13}$$

is injective. Composing on the right with the evident map, it suffices to show the maps

$$\begin{aligned} &\mathrm{Sym}^i(H^0(J^\vee, \Omega^1_{J^\vee/K})^*) \otimes H^0(G_0, \mathcal{O}_{G_0}) \\ &\qquad \to H^1(G_0, \mathcal{O}_{G_0}) \otimes \mathrm{Sym}^{i-1}(H^0(J^\vee, \Omega^1_{J^\vee/K})^*) \end{aligned} \tag{2.14}$$

are injective. But

$$\begin{aligned} H^0(J^\vee, \Omega^1_{J^\vee/K})^* \otimes H^0(G_0, \mathcal{O}_{G_0}) &\cong H^1(J, \mathcal{O}_J) \otimes H^0(G_0, \mathcal{O}_{G_0}) \\ &\cong H^1(G_0, \mathcal{O}_{G_0}) \end{aligned} \tag{2.15}$$

and the map in (2.14) is the map $x^i \mapsto x \otimes x^{i-1}$, which is injective. $\square$

Lemma 2.4. *Let $G = J_\mathcal{D}$ be a generalized jacobian as above. Then there exists a commutative affine algebraic group $\mathbb{G}$ over K and a map $\psi : G \to \mathbb{G}$ such that*

$$\psi^* : H^0(\mathcal{O}_\mathbb{G}) \cong H^0(\mathcal{O}_G). \tag{2.16}$$

Proof. Take $\mathbb{G} = \mathrm{Spec}\,(H^0(G, \mathcal{O}_G))$. $\square$

Lemma 2.5. *Let A be the coordinate ring of a commutative affine algebraic group H over a field K of characteristic 0. Corresponding to the simplicial algebraic group BH, one has a complex*

$$A \xrightarrow{\mu^* - p_1^* - p_2^*} A \otimes_K A \xrightarrow{p_{23}^* - \mu_{12} \otimes p_3^* + p_1^* \otimes \mu_{23}^* - p_{12}^*} A \otimes A \otimes A \tag{2.17}$$

This complex is exact at the middle term.

Proof. By proposition 4 on p. 168 of [10], the cohomology in the middle is a subgroup of the group of extensions $Ext(H, \mathbb{G}_a)$. (Note, $A = Map(H, \mathbb{G}_a)$.) By the classification of commutative algebraic groups in characteristic 0, this ext group vanishes (cf [10], pp. 170-172). □

We write Ω^1_G (resp. $\Omega^1_{G/K}$) for the sheaf of 1-forms relative to k (resp. K). Now we would like to define invariant bundles, connections, differential forms, cohomology classes of $\mathcal{O}_G$.

Definition 2.6.
1. *A rank one bundle* $L \in H^1(G, \mathcal{O}^*_G)$ *is called invariant if* $\mu^* L = p_1^* L \otimes p_2^* L \in H^1(G \times G, \mathcal{O}^*_{G\times G})$, *where* $\mu : G\times G \to G$ *is the multiplication and* $p_i : G \times G \to G$ *are the projections.*
2. *A global* 1*-form* $\eta \in \Gamma(G, \Omega^1_G)$ *is called invariant if* $\eta(0) = 0 \in \Omega^1_K$ *and* $\mu^*\eta = p_1^*\eta + p_2^*\eta \in \Gamma(G \times G, \Omega^1_{G\times G})$.
3. *A rank one bundle with a connection* $(L, \nabla) \in \mathbb{H}^1(G, \mathcal{O}^*_G \to \Omega^1_G)$ *is called invariant if* $(L, \nabla)|\{0\} = 0 \in \mathbb{H}^1(\mathrm{Spec}\, K, \mathcal{O}^*_{\mathrm{Spec}\, K} \to \Omega^1_{\mathrm{Spec}\, K}) = \Omega^1_K / d\log K^*$ *and* $\mu^*(L, \nabla) = p_1^*(L, \nabla) \otimes p_2^*(L, \nabla) \in \mathbb{H}^1(G \times G, \mathcal{O}^*_{G\times G} \to \Omega^1_{G\times G})$.
4. *A class* $s \in H^i(G, \mathcal{O}_G)$ *is called invariant if* $s|\{0\} = 0$ *and* $\mu^* s = p_1^* s + p_2^* s$ *in* $H^i(G \times G, \mathcal{O}_{G\times G})$.

We denote by $H^1(G, \mathcal{O}^*_G)^{\mathrm{inv}}$, $\Gamma(G, \Omega^1_G)^{\mathrm{inv}}$, $\mathbb{H}^1(G, \mathcal{O}^*_G \to \Omega^1_G)^{\mathrm{inv}}$, *and* $H^i(G, \mathcal{O}_G)^{\mathrm{inv}}$ *the corresponding groups of invariant bundles, forms, connections and classes. One defines similarly the groups of relative invariant forms* $H^0(G, \Omega^1_{G/K})^{\mathrm{inv}}$ *and relative invariant connections* $\mathbb{H}^1(G, \mathcal{O}^*_G \to \Omega^1_{G/K})^{\mathrm{inv}}$ *without condition on the restriction to the zero section, and observe that the natural map* $\Omega^1_G \to \Omega^1_{G/K}$ *takes global invariant groups to relative invariant groups.*

Remark 2.7. *In the above definitions, we could have defined a weaker notion of invariance by allowing constant elements. We adopt here the rigidification at the origin, keeping in mind that without this condition, the corresponding groups obtained are a direct sum of the ones obtained with the rigidification and the value of the group on the zero section. Notice, for example, that with our definition, nonzero constant functions are not invariant! Of course, for relative objects, there is no distinction.*

Lemma 2.8. *Let* $G = J_{\mathcal{D}}$ *be a generalized Jacobian as above. Let* $\tau \in \Gamma(G, \Omega^1_{G/K})^{\mathrm{inv}}$ *be an invariant relative* 1*-form on* G, *and assume* τ *lifts to an absolute global form. Then* τ *lifts to an invariant absolute form on* G.

Proof. Let $\eta \in \Gamma(G, \Omega^1_G)$ be an absolute lifting. Replacing η with $\eta - \eta(0)$ we may assume $\eta(0) \in \Omega^1_K$ vanishes. Then

$$(\mu^* - p_1^* - p_2^*)(\eta) \in H^0(\mathcal{O}_{G\times_K G}) \otimes \Omega^1_K \tag{2.18}$$

vanishes in $H^0(\mathcal{O}_{G\times_K G\times_K G}) \otimes \Omega^1_K$. Let $\psi : G \to \mathbb{G}$ be as in lemma 2.4, so $\psi^* : A := H^0(\mathcal{O}_{\mathbb{G}}) \cong H^0(\mathcal{O}_G)$. The previous lemma implies there exists $\sigma \in H^0(\mathcal{O}_G) \otimes \Omega^1_K$ with $(\mu^* - p_1^* - p_2^*)(\sigma) = (\mu^* - p_1^* - p_2^*)(\eta)$. Then $\eta - \sigma$ is the desired invariant absolute form. $\square$

We next need to relate connections on the curve with invariant connections on the generalized Jacobian. Here $G = J_{\mathcal{D}}$ with $\mathcal{D} = \sum m_i c_i$. Also,

$$D := \sum c_i; \quad \mathcal{D}' := \mathcal{D} - D. \tag{2.19}$$

We assume $D \neq \emptyset$.

First, rather briefly, we consider the question of what poles invariant forms on G have when pulled back to $C - D$ via the jacobian map $C - D \to G$ (defined once we have a basepoint in $C - D$). For simplicity, we continue to assume the c_i are defined over K. Consider the diagram:

$$\begin{array}{ccc} G \twoheadrightarrow G_0 & \twoheadrightarrow & J \\ \uparrow i & & \uparrow i' \\ C - D & \stackrel{j}{\hookrightarrow} & C \end{array} \tag{2.20}$$

We want to compute the pullbacks $i^* H^0(\Omega^1_G)^{inv}$ (resp. $i^* H^0(\Omega^1_{G/K})^{inv}$) in $H^0(C, \Omega^1_C(*D))$ (resp. in $H^0(C, \Omega^1_{C/K}(*D))$) .

Pulling back G via i' we get a torseur

$$i'^* G \xrightarrow{p} C \tag{2.21}$$

under the group $\mathcal{O}^*_{\mathcal{D}}/\mathbb{G}_m = \prod \mathcal{O}^*_{m_\ell c_\ell}/\mathbb{G}_m$. Fix ℓ and let $R = k[[t_\ell]] \subset M = k((t_\ell))$ where t_ℓ is a formal parameter at c_ℓ. Fix a splitting of the torseur

$$G_R \cong \mathcal{O}^*_{\mathcal{D}}/\mathbb{G}_m \times \mathrm{Spec}\,(R). \tag{2.22}$$

Let $c_0 \in (C - D)(K)$ be the base point used to define the map $i : C - D \to G$. Let $\mathcal{L}$ on $C \times C$ be a line bundle with $\mathcal{L}|_{\{c\}\times C} \cong \mathcal{O}(c - c_0)$. Note one has

$$\mathcal{O}_{C\times C} \leftarrow p_2^* \mathcal{O}(-c_0) \to \mathcal{L} \tag{2.23}$$

and these maps are isomorphisms on

$$\mathrm{Spec}\,(M) \times \mathrm{Spec}\,(\mathcal{O}_{C,D}) \subset \mathrm{Spec}\,(M) \times C \tag{2.24}$$

Corresponding to $\mathcal{L}|_{\mathrm{Spec}\,(M)\times C}$ and the above trivialization, one gets a map $u : \mathrm{Spec}\,(M) \to G$. With respect to the above splitting, we view u as an element

(2.25)
$$\prod_i (u_{i0} + u_{i1}t_i + \ldots + u_{i,m_i-1}t_i^{m_i-1}) \in \prod_i (\mathcal{O}_{m_i c_i} \otimes_K M)^* \mod M^*$$

As described in [10] VII 4, 21, the local shape of u around c_ℓ is given by taking the rational function $(s_\ell - t_\ell)^{-1}$, where the local coordinates around (c_ℓ, c_ℓ) in $C\times C$ is (s_ℓ, t_ℓ), and considering it as a unit in $\mathcal{O}_{m_\ell c_\ell} \otimes M \cong M[t_\ell]/ < t_\ell^{m_\ell} >$. (We change notation so s_ℓ is the local parameter in $R \subset M$.) Since u is well defined and non-vanishing in c_i for $i \neq \ell$, we have (2.24) that

$$u_{ij} \in R, u_{i0} \in R^* \text{ if } i \neq \ell,\ \mathrm{ord}(u_{\ell 0}) = 1 \tag{2.26}$$

The pullbacks to Spec (M) of the invariant relative differential forms on G are given by the pullback of invariant relative forms on J together with the coefficients of powers of the $T_i \mod T_i^{m_i}$ in the formal expression

$$\begin{aligned}(2.27)\quad \sum_i (u_{i0} + u_{i1}T_i + \ldots + u_{i,m_i-1}T_i^{m_i-1})^{-1} \times \\ (du_{i0} + du_{i1}T_i + \ldots + du_{i,m_i-1}T_i^{m_i-1}) \\ = \sum_i \tau_{i0} + \ldots + \tau_{i,m_i-1}T_i^{m_i-1}.\end{aligned}$$

Then (2.26) implies that

$$\begin{aligned}&\tau_{ij} \in \Omega^1_R \text{ for } i \neq \ell \\ (2.28)\quad &\tau_{\ell j} \in \Omega^1_R\langle D\rangle(jD_\ell) - \Omega^1_R\langle D\rangle((j-1)D_\ell).\end{aligned}$$

Here we denote by $\Omega^1_C\langle D\rangle$ the sheaf of absolute differential forms of degree 1 with logarithmic poles along D. (See formula 3.59 for a more precise computation).

These are not all the absolutely invariant forms, however. One also has forms pulled back from J, but these are regular along D. Finally, from lemma 2.8 one has an exact sequence

$$\begin{aligned}(2.29)\quad 0 \to H^0(\mathcal{O}_G)^{\mathrm{inv}} \otimes \Omega^1_K \to H^0(G, \Omega^1_G)^{\mathrm{inv}} \to H^0(G, \Omega^1_{G/K})^{\mathrm{inv}} \\ \to H^1(\mathcal{O}_G)^{\mathrm{inv}} \otimes \Omega^1_K\end{aligned}$$

It shows one must consider invariant forms in $H^0(\mathcal{O}_G)^{\mathrm{inv}} \otimes \Omega^1_K$. We will see in the proof of proposition 2.13 below that these map to $\Omega^1_G(\mathcal{D}')$.

In sum, the above discussion shows that the maps in the following proposition are defined.

Proposition 2.9. *Pullback gives isomorphisms*

$$H^0(\Omega^1_{G/K})^{inv} \xrightarrow{\cong} H^0(C, \Omega^1_{C/K}(\mathcal{D})) \tag{2.30}$$

$$H^0(\Omega^1_{G})^{inv} \xrightarrow{\cong} H^0(C, \Omega^1_{C}\langle D\rangle(\mathcal{D}')) \tag{2.31}$$

Proof. Pullback on invariant relative forms is injective, because G is generated by the image of $C - D$. It follows by dimension count that the first arrow (2.30) above is an isomorphism. For the absolute forms we may consider the diagram

(2.32)
$$\begin{array}{ccccccccc} 0 & \to & H^0(\mathcal{O}_G)^{\rm inv} \otimes \Omega^1_K & \to & H^0(\Omega^1_G)^{\rm inv} & \to & H^0(\Omega^1_{G/K})^{\rm inv} & \to & H^1(\mathcal{O}_G)^{\rm inv} \otimes \Omega^1_K \\ & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong \\ 0 & \to & H^0(\mathcal{O}_C(\mathcal{D}')) \otimes \Omega^1_K & \to & H^0(\Omega^1_C\langle D\rangle(\mathcal{D}')) & \to & H^0(\Omega^1_{C/K}(\mathcal{D})) & \to & H^1(\mathcal{O}_C(\mathcal{D}')) \otimes \Omega^1_K \end{array}$$

The left and right hand vertical arrows are shown to be isomorphisms in the proof of proposition 2.13. Hence the isomorphism on invariant relative forms implies the isomorphism (2.31) on invariant absolute forms. □

We now consider invariant connections on line bundles on G.

Lemma 2.10. *Assume the toric subquotient T of G has trivial Picard group (e.g. T split). Then the map $\mathbb{H}^1(G, \mathcal{O}^*_G \to \Omega^1_{G/K})^{\rm inv} \to H^1(\mathcal{O}^*_G)^{\rm inv}$ is surjective.*

Proof. This follows because

$$\mathbb{H}^1(J, \mathcal{O}^*_J \to \Omega^1_{J/K})^{\rm inv} \twoheadrightarrow H^1(\mathcal{O}^*_J)^{\rm inv} \twoheadrightarrow H^1(\mathcal{O}^*_G)^{\rm inv}. \tag{2.33}$$

The second arrow is surjective because we have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(\mathcal{O}^*_J)^{\rm inv} & \longrightarrow & H^1(\mathcal{O}^*_J) & \longrightarrow & H^2_{DR}(J/K) \\ & & \downarrow & & \downarrow{\scriptstyle surj.} & & \downarrow{\scriptstyle inj.} \\ 0 & \longrightarrow & H^1(\mathcal{O}^*_G)^{\rm inv} & \xrightarrow[a]{} & H^1(\mathcal{O}^*_G) & \xrightarrow[b]{} & H^2_{DR}(G/K) \end{array} \tag{2.34}$$

The bottom row is not a priori exact, but $b \circ a = 0$ (because $(N\delta)^*$ acts by N^2 on $H^2_{DR}(G/K)$.) The middle vertical arrow is onto e.g. because the Picard group of the generic fibre of $G \to J$ is zero. Indeed, $G \to J$ is rationally split, and the kernel has trivial Picard group by hypothesis. (Since the function field of the generic fibre equals the function field of G, any divisor on G can be moved by rational equivalence to avoid the generic fibre, i.e. to be a pullback from the base.)

Finally, the right hand vertical arrow is injective because, after making a base change $K \subset \mathbb{C}$, one can think of G and J as quotients of vector spaces by lattices, and the map on lattices is surjective $\otimes\mathbb{Q}$. $\square$

Lemma 2.11. *Let $a \in \mathbb{H}^1(G, \mathcal{O}_G^* \to \Omega^1_{G/K})^{\mathrm{inv}}$ be an invariant connection on a line bundle on the generalized jacobian G. Suppose a lifts to an absolute connection $b' \in \mathbb{H}^1(G, \mathcal{O}_G^* \to \Omega^1_G)$. Then a lifts to an invariant absolute connection $b \in \mathbb{H}^1(G, \mathcal{O}_G^* \to \Omega^1_G)^{\mathrm{inv}}$.*

Proof. One has as in (2.17)

$$(\mu^* - p_1^* - p_2^*)(b') \in \ \mathrm{Im}\ H^0(\mathcal{O}_{G\times G}) \otimes \Omega^1_K \subset \mathbb{H}^1(G\times G, \mathcal{O}^*_{G\times G} \to \Omega^1_{G\times G}) \tag{2.35}$$

Now

$$\begin{gathered}\mathrm{Im}\ H^0(\mathcal{O}_{G\times G}) \otimes \Omega^1_K = \\ H^0(\mathcal{O}_{G\times G}) \otimes \Omega^1_K / d\log \mathrm{Ker}\ \{H^0(\mathcal{O}^*_{G\times G}) \to H^0(\Omega^1_{G\times G/K})\} = \\ H^0(\mathcal{O}_{G\times G}) \otimes (\Omega^1_K / d\log K^*).\end{gathered}$$

Exactness of the sequence in (2.17) implies that there exists an element $x \in H^0(G, \mathcal{O} \otimes \Omega^1_K)$ with $(\mu^* - p_1^* - p_2^*)(x) = (\mu^* - p_1^* - p_2^*)(b')$. Take $b = b' - x$. $\square$

Proposition 2.12. *One has an exact sequence*

$$0 \to H^0(\mathcal{O}_G)^{\mathrm{inv}} \otimes \Omega^1_K \to \mathbb{H}^1(G, \mathcal{O}_G^* \to \Omega^1_G)^{\mathrm{inv}} \to \mathbb{H}^1(G, \mathcal{O}_G^* \to \Omega^1_{G/K})^{\mathrm{inv}} \to H^1(\mathcal{O}_G)^{\mathrm{inv}} \otimes \Omega^1_K \tag{2.36}$$

Proof. This is immediate from the lemma. $\square$

Recall our notation. C is a smooth, projective, geometrically connected curve over K. $G = J_{\mathcal{D}}$ with $\mathcal{D} = \sum m_i c_i$, $D = \sum c_i$, $\mathcal{D}' = \mathcal{D} - D$.

Proposition 2.13. *There exists a diagram of exact sequences, with vertical arrows isomorphisms:*

$$\begin{array}{ccccc} 0 & \to & H^0(\mathcal{O}_G)^{\mathrm{inv}} \otimes (\Omega^1_K/d\log K^*) & \to & \mathbb{H}^1(G, \mathcal{O}_G^* \to \Omega^1_G)^{\mathrm{inv}} \\ & & \downarrow e & & \downarrow \\ 0 & \to & \Big(H^0(\mathcal{O}_C(\mathcal{D}'))/H^0(\mathcal{O}_C)\Big) \otimes (\Omega^1_K/d\log K^*) & \to & \frac{\mathbb{H}^1(C, j_*(\mathcal{O}^*_{C-D}) \to \Omega^1_C\langle D\rangle(\mathcal{D}'))}{(\Omega^1_K/K^\times)} \end{array}$$

(2.37)

$$\begin{array}{ccccc} \to & \mathbb{H}^1(G, \mathcal{O}_G^* \to \Omega^1_{G/K})^{\mathrm{inv}} & \to & H^1(\mathcal{O}_G)^{\mathrm{inv}} \otimes \Omega^1_K \\ & \downarrow & & \downarrow h \\ \to & \mathbb{H}^1\Big(C, j_*(\mathcal{O}^*_{C-D}) \to \Omega^1_{C/K}(\mathcal{D})\Big) & \to & H^1(\mathcal{O}_C(\mathcal{D}')) \otimes \Omega^1_K \end{array}$$

Proof. The first step is to compute $H^i(\mathcal{O}_G)^{\mathrm{inv}}$ for $i = 0, 1$. Let W be a finite dimensional K-vector space, and suppose $G = J_{\mathcal{D}}$ is a vectorial extension

$$0 \to W \otimes \mathbb{G}_a \to G \xrightarrow{p} G_0 \to 0 \tag{2.38}$$

We know by lemma 2.1 that this sequence pulls back from an extension of J by $W \otimes \mathbb{G}_a$. Let

$$0 \to \mathcal{O}_J \to fil_1 \to W^* \otimes \mathcal{O}_J \to 0 \tag{2.39}$$

be the exact sequence of functions of filtration degree ≤ 1 as in lemma 2.3, and let $\partial : W^* \to H^1(\mathcal{O}_J)$ be the boundary map in cohomology.

Lemma 2.14. *We have*

$$H^0(\mathcal{O}_G)^{\mathrm{inv}} \cong \ker(\partial : W^* \to H^1(\mathcal{O}_J)) \tag{2.40}$$

$$H^1(\mathcal{O}_G)^{\mathrm{inv}} \cong H^1(\mathcal{O}_J)/\partial(W^*). \tag{2.41}$$

proof of lemma. One has a filtration $fil.p_*\mathcal{O}_G$ with $fil_0 = \mathcal{O}_{G_0}$ and $gr_r = \mathrm{Sym}^r(W^*) \otimes \mathcal{O}_{G_0}$. The corresponding spectral sequence looks like

(2.42)
$$E_1^{pq} = H^{p+q}(G_0, gr_{-p}) = H^{p+q}(G_0, \mathrm{Sym}^{-p}(W^*) \otimes \mathcal{O}_{G_0}) \Rightarrow H^{p+q}(\mathcal{O}_G).$$

Let

$$0 \to H_0 \to G_0 \to S \to 0 \tag{2.43}$$

be as in lemma 2.2, so S is the maximal quotient torus of G_0.

The equation (2.6) identifies $H^i(G_0, \mathcal{O}_{G_0})$ with $H^i(J \times S, \mathcal{O}_{J\times S})$, and the invariance condition might be looked at on $J \times S$. Let us write $H^0(S, \mathcal{O}_S) = K \oplus V, f \mapsto f(0) \oplus (f - f(0))$, where V consists of the regular functions which vanish at $\{0\} \in S$. Then $H^i(G_0, \mathcal{O}_{G_0})^{\mathrm{inv}} = H^i(J, \mathcal{O}_J)^{\mathrm{inv}} \oplus (H^i(J, \mathcal{O}_J) \otimes V)^{\mathrm{inv}}$. Thus if a class $F = \sum \varphi_f \otimes f \in H^i(\mathcal{O}_J) \otimes V$ is invariant, where $\varphi_f \in H^i(J, \mathcal{O}_J)$ and the $f \in V$ are linearly independent over K, then

$$(\mu^* - p_1^* - p_2^*)(F)|(J \times J \times \{0\} \times S) =$$
$$\sum (\mu^* - p_2^*)(\varphi_f) \otimes f = 0$$

thus $\mu^*\varphi_f = p_i^*\varphi_f$ and $\mu^*\varphi_f|\{0\} \times J = \varphi_f = \varphi_f|\{0\}$. So for $i \geq 1$, this implies that $\varphi_f = 0$ and for $i = 0$ this implies that $F \in H^0(\mathcal{O}_S)^{\mathrm{inv}} = 0$.

In short:

(2.44)
$$H^*(\mathcal{O}_{G_0})^{\mathrm{inv}} \cong \Big(H^*(\mathcal{O}_J) \otimes H^0(\mathcal{O}_S)\Big)^{\mathrm{inv}} \cong H^*(\mathcal{O}_{H_0})^{\mathrm{inv}} \cong H^*(\mathcal{O}_J)^{\mathrm{inv}}$$

Thus, it suffices to prove the lemma with G_0 replaced by H_0, so we may assume the quotient torus $S = (0)$. Since in this case $H^i(\mathcal{O}_{G_0}) \cong \wedge^i H^1(\mathcal{O}_J)$, one sees that pullback under the multiplication by N map, $N\delta : G \to G$ acts on E_1^{pq} by multiplication by N^q. It follows that the spectral sequence (2.42) degenerates at E_2. In particular the eigenspace where $N\delta^*$ acts by multiplication by N on $H^1(\mathcal{O}_G)$ is

$$H^1(\mathcal{O}_J)/\partial(W^*) \cong E_2^{0,1} \cong E_\infty^{0,1} \hookrightarrow H^1(\mathcal{O}_G). \tag{2.45}$$

Note that as a quotient of $H^1(\mathcal{O}_J)$ the space $E_\infty^{0,1}$ is clearly invariant. Conversely, let $\Delta : G \to G \times G$ be the diagonal. Since $\mu \circ \Delta = 2\delta$ it follows that for $a \in H^1(\mathcal{O}_G)^{\mathrm{inv}}$ we have

$$(2\delta)^*(a) = \Delta^*\mu^*(a) = \Delta^*(p_1^*(a) + p_2^*(a)) = 2a \tag{2.46}$$

so necessarily $a \in E_\infty^{0,1}$, proving (2.41). A similar argument on $E_\infty^{-1,1}$ proves (2.40).

We remark here that $H^0(J, fil_1)$ is in a natural way a subspace of the regular functions on G, and (2.40) takes the quotient of this by $H^0(J, \mathcal{O}_J) = K$. This is because we have forced the rigidification condition in the definition 2.6. $\square$

We return to the proof of proposition 2.13. The exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(\mathcal{D}') \to \mathcal{O}_C(\mathcal{D}')/\mathcal{O}_C \to 0 \tag{2.47}$$

defines a map

$$\psi : W^* := H^0(\mathcal{O}_C(\mathcal{D}')/\mathcal{O}_C) \to H^1(\mathcal{O}_C). \tag{2.48}$$

By lemma 2.1, as a group extension of G_0, the group G is defined by a unique map from $H^0(C, \Omega^1_{C/K})$ to a vector space. We claim that this map is the dual of ψ. To see this, one identifies J and $J^\vee$. Then it is well know that the universal vectorextension on $J^\vee$ is

$$0 \to H^0(C, \Omega^1_{C/K}) \to \mathbb{H}^1(C, \mathcal{O}_C^* \to \Omega^1_{C/K}) \to \mathrm{Pic}^0(C) \to 0$$

inducing the universal vectorextension

$$0 \to H^0(C, \Omega^1_{C/K}) \to \mathbb{H}^1(C, \mathcal{O}_{C,D}^* \to \Omega^1_{C/K}) \to \mathrm{Pic}^0(C, D) \to 0$$

on

$$\mathrm{Pic}^0(C, D) := \mathrm{Ker}\Big(H^1(C, \mathcal{O}_{C,D}^*) \to H^1(C, \Omega^1_{C/K})\Big)$$

where

$$\mathcal{O}_{C,Z}^* := \mathrm{Ker}(\mathcal{O}_C^* \to \mathcal{O}_Z^*)$$

for any subscheme $Z \subset C$. The map of complexes

$$a : \{\mathcal{O}_{C,\mathcal{D}}^* \to 0\} \to \{\mathcal{O}_{C,D}^* \to \Omega^1_{C/K}|_{\mathcal{D}'}\}$$

induces an isomorphism on $\mathbb{H}^1$. Indeed, a sends the exact sequence

$$0 \to H^0(C, (1+\mathcal{O}_{\mathcal{D}'}(-D))) \to H^1(C, \mathcal{O}^*_{C,\mathcal{D}}) \to H^1(C, \mathcal{O}^*_{C,D}) \to 0$$

to the exact sequence

$$0 \to H^0(C, \Omega^1_{C/K}|_{\mathcal{D}'}) \to \mathbb{H}^1(C, \mathcal{O}^*_{C,D} \to \Omega^1_{C/K}|_{\mathcal{D}'}) \to H^1(C, \mathcal{O}^*_{C,D}) \to 0,$$

so one has just to see that

$$d\log : H^0(C, (1+\mathcal{O}_{\mathcal{D}'}(-D))) \to H^0(C, \Omega^1_{C/K}|_{\mathcal{D}'})$$

is an isomorphism. But $H^0(C, \mathcal{O}_{\mathcal{D}'}(-D)) \cong H^0(C, (1+\mathcal{O}_{\mathcal{D}'}(-D)))$ via the exponential map and the quasiisomorphism [4]

$$\{\mathcal{O}_C(-\mathcal{D}) \to \Omega^1_{C/K}(-\mathcal{D}')\} \to \{\mathcal{O}_C(-D) \to \Omega^1_{C/K}\}$$

allows to conclude.

Define a bundle $\mathcal{E}$ on C by pullback

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & W^* \otimes \mathcal{O}_C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}_C \arrow[r] & \mathcal{O}(\mathcal{D}') \arrow[r] & \mathcal{O}_C(\mathcal{D}')/\mathcal{O}_C \arrow[r] & 0 \end{tikzcd} \tag{2.49}$$

Because of the isomorphism $H^1(\mathcal{O}_J) \cong H^1(\mathcal{O}_C)$, the top row of the diagram (2.49) pulls back uniquely from an extension of $W^* \otimes \mathcal{O}_J$ by $\mathcal{O}_J$. There is a unique vectorial extension

$$0 \to W \otimes \mathbb{G}_a \to H \xrightarrow{r} J \to 0 \tag{2.50}$$

such that the above extension of vector bundles coincides with

$$0 \to \mathcal{O}_J \to fil_1 r_* \mathcal{O}_H \to W^* \otimes \mathcal{O}_J \to 0 \tag{2.51}$$

From this we get a diagram (defining t and u. Here $i : C \hookrightarrow J$)

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_J \arrow[r] \arrow[d] & fil_1 r_* \mathcal{O}_H \arrow[r] \arrow[d, "t"] & W^* \otimes \mathcal{O}_J \arrow[r] \arrow[d, "u"] & 0 \\ 0 \arrow[r] & \mathcal{O}_C \arrow[r] & \mathcal{O}_C(\mathcal{D}') \arrow[r] & \mathcal{O}_C(\mathcal{D}')/\mathcal{O}_C \arrow[r] & 0 \end{tikzcd} \tag{2.52}$$

We get a diagram with exact rows

(2.53)

$$\begin{array}{ccccccccc} 0 & \to & H^0(\mathcal{O}_J) & \to & H^0(fil_1 r_* \mathcal{O}_H) & \to & H^0(W^* \otimes \mathcal{O}_J) & \xrightarrow{\partial} & H^1(\mathcal{O}_J) & \to \\ & & \downarrow\cong & & \downarrow t & & \downarrow\cong & & \downarrow\cong & \\ 0 & \to & H^0(\mathcal{O}_C) & \to & H^0(\mathcal{O}_C(\mathcal{D}')) & \to & H^0(\mathcal{O}_C(\mathcal{D}')/\mathcal{O}_C) & \to & H^1(\mathcal{O}_C) & \to \end{array}$$

$$\begin{array}{ccc} H^1(\mathcal{O}_H)^{\mathrm{inv}} & \to & 0 \\ \downarrow v & & \\ H^1(\mathcal{O}_C(\mathcal{D}')) & \to & 0 \end{array}$$

The diagram (2.53) gives isomorphisms

$$e : H^0(\mathcal{O}_G)^{\mathrm{inv}} \cong \ker(\partial) \cong \mathrm{Ker}(H^0(\mathcal{O}_C(\mathcal{D}')/\mathcal{O}_C) \to H^1(\mathcal{O}_C)) \tag{2.54}$$

$$h : H^1(\mathcal{O}_G)^{\mathrm{inv}} \cong H^1(\mathcal{O}_C(\mathcal{D}')). \tag{2.55}$$

These are two of the desired arrows for the diagram in the proposition.

Lemma 2.15. *The natural map on relative connections*

$$\mathbb{H}^1(G, \mathcal{O}_G^* \to \Omega^1_{G/K})^{\mathrm{inv}} \to \mathbb{H}^1\Big(C, j_*(\mathcal{O}^*_{C-D}) \to \Omega^1_{C/K}(\mathcal{D})\Big) \tag{2.56}$$

is an isomorphism.

proof of lemma. We note the following facts:

1. $H^1(G, \Omega^1_{G/K})^{\mathrm{inv}} = \Big(H^0(G, \Omega^1_{G/K})^{\mathrm{inv}} \otimes H^1(\mathcal{O}_G)\Big)^{\mathrm{inv}} = (0)$.
2. $H^1(\mathcal{O}_G^*)^{\mathrm{inv}} \cong H^1(\mathcal{O}^*_{C-D})$. Indeed, as remarked in the proof of lemma 2.10 one has $J(K) \twoheadrightarrow H^1(\mathcal{O}_G^*)^{\mathrm{inv}}$. One checks that the kernel is generated by divisors of degree 0 supported on D.
3. $H^0(\mathcal{O}_G^*)/\text{consts.} \cong H^0(\mathcal{O}_G^*)^{\mathrm{inv}} \cong H^0(\mathcal{O}^*_{C-D})/\text{consts.}$.
4. $$\Big(H^0(\Omega^1_{G/K})/d\log(H^0(\mathcal{O}_G^*))\Big)^{\mathrm{inv}} = H^0(\Omega^1_{G/K})^{\mathrm{inv}}/d\log(H^0(\mathcal{O}_G^*)^{\mathrm{inv}})$$

   (This is seen by noting $H^0(\mathcal{O}_G^*)^{\mathrm{inv}} = \mathrm{Hom}(G, \mathbb{G}_m)$, so one has a homomorphism $\psi : G \to \mathbb{G}_m^r$ such that ψ^* is an isomorphism on global units modulo constants. The assertion then reduces to the case $G = \mathbb{G}_m^r$, which is easy.)

We build a diagram

(2.57)

$$\begin{array}{ccccccccc} 0 & \to & \frac{H^0(\Omega^1_{G/K})^{\mathrm{inv}}}{H^0(\mathcal{O}_G^*)^{\mathrm{inv}}} & \to & \mathbb{H}^1(\mathcal{O}_G^* \to \Omega^1_{G/K})^{\mathrm{inv}} & \to & H^1(\mathcal{O}_G^*)^{\mathrm{inv}} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \frac{H^0(\Omega^1_{C/K}(\mathcal{D}))}{H^0(\mathcal{O}^*_{C-D})} & \to & \mathbb{H}^1\Big(C, j_*(\mathcal{O}^*_{C-D}) \to \Omega^1_{C/K}(\mathcal{D})\Big) & \to & H^1(C-D, \mathcal{O}^*) & \to & 0 \end{array}$$

Since the left and right hand arrows are isomorphisms, it follows that the central arrow is as well, proving the lemma. □

The assertions of the proposition follow easily from the lemma. □

Finally, we need an analogous result for integrable connections. More precisely, we consider a slightly weaker condition.

Definition 2.16. *Let X be a variety over K. A connection $\nabla : E \to E \otimes \Omega^1_X \otimes K(X)$ (so possibly with poles) is said to have vertical curvature if the curvature*

$$\nabla^2 : E \to E \otimes \Omega^2_X \otimes K(X) \tag{2.58}$$

has values in the subsheaf $E \otimes \Omega^2_K \otimes K(X) \subset E \otimes \Omega^2_X \otimes K(X)$. The group of line bundles with vertical curvature will be denoted

$$\mathbb{H}^1(X, \mathcal{O}_X^* \to \Omega^1_X)^{\mathrm{vert}}$$

and similarly for invariant line bundles with vertical curvature

$$\mathbb{H}^1(X, \mathcal{O}_G^* \to \Omega^1_G)^{\mathrm{inv,vert}} .$$

Proposition 2.17. *With notation as above, we have isomorphisms*

$$\mathbb{H}^1(G, \mathcal{O}_G^* \to \Omega^1_{G/K})^{\mathrm{inv,vert}} = \mathbb{H}^1(C, j_*\mathcal{O}^*_{C-D} \to \Omega^1_{C/K}(\mathcal{D}))^{\mathrm{vert}} \tag{2.59}$$

$$\mathbb{H}^1(G, \mathcal{O}_G^* \to \Omega^1_G)^{\mathrm{inv,vert}} \cong \frac{\mathbb{H}^1(C, j_*\mathcal{O}^*_{C-D} \to \Omega^1_C\langle D\rangle(\mathcal{D}'))^{\mathrm{vert}}}{\Omega^1_K/K^\times} \tag{2.60}$$

Proof. For example, in the absolute case, the curvature of a line bundle with invariant absolute connection on G is a section $\eta \in H^0(\Omega^2_G/\Omega^2_K \otimes \mathcal{O}_G)$ satisfying $\mu^*(\eta) = p_1^*(\eta)+p_2^*(\eta)$. It is easy to see that such a section lies in the subsheaf $\Omega^1_{G/K} \otimes \Omega^1_K$. The isomorphism (2.60) follows from proposition 2.13 and the fact that pullback to C of invariant forms is injective by (2.30). The case of relative forms is similar and is left for the reader. □

3. The Geometric Setup

We continue to work with a curve C/K and a line bundle L on C of degree 0. Let $\nabla_{/K} : L \to L \otimes \Omega^1_{C/K}(\mathcal{D})$, where $\mathcal{D} = \sum m_i c_i$. As in the previous section, write $D = \sum c_i$ and $\mathcal{D}' = \mathcal{D} - D$.

Lemma 3.1. *Assume $\nabla_{/K}|C - D$ lifts to an absolute integrable connection $\nabla' : L_{C-D} \to L_{C-D} \otimes \Omega^1_{C-D/k}$. Then ∇' extends to an absolute integrable connection*

$$\nabla : L \to L \otimes \Omega^1_{C/k}\langle D\rangle(\mathcal{D}'). \tag{3.1}$$

The notation $\langle D\rangle$ refers to log poles at D as in (1.4).

Proof. Let e be a basis for L at c a point with multiplicity $m \geq 1$ in $\mathcal{D}$, and let x be a local parameter at c on C. Write

(3.2)
$$\nabla'(e) = A(x)dx + \sum_i B_i(x)d\tau_i; \quad d\tau_i \text{ basis in } \Omega^1_K,\ x^m A(x) \in \mathcal{O}_{C,c}.$$

We must show $x^{m-1}B_i(x)$ is regular at c. But integrability of ∇' implies that $\partial A/\partial\tau_i = \partial B_i/\partial x$, from which the assertion is clear. □

We know from proposition 2.17 that the restriction to $C - D$ of an integrable absolute connection of the form (3.1) pulls back from a unique invariant integrable absolute connection $\mathcal{L} \to \mathcal{L} \otimes \Omega^1_G$ on $G = J_{\mathcal{D}}$. More precisely, we fix a basepoint $c_0 \in (C-D)(K)$ and normalize our connection (3.1) to be trivial at the basepoint by tensoring with a pullback from $\mathrm{Spec}\,(K)$.

We consider now the basic geometric picture of Deligne [3]

$$\pi : \mathrm{Sym}^N(C-D) \to G_N; \quad N = 2g-2+\sum_i m_i \tag{3.3}$$

where G_N is the $J_{\mathcal{D}}$-torseur of degree N line bundles trivialized along $\mathcal{D}$ and $\pi(\sum z_i) = \mathcal{O}(\sum z_i)$ with trivialization given by restricting to $\mathcal{D}$ the canonical (upto scalar in K) map $\mathcal{O}_C \to \mathcal{O}_C(\sum z_i)$. Note that $N = \deg(\Omega^1_{C/K}(\mathcal{D}))$ and $\dim G = g-1+\sum m_i = N-g+1$. (Recall we assume $\mathcal{D} \neq \emptyset$.) We identify $G_N \cong G$ by sending the point $[\mathcal{O}(Nc_0)] \mapsto 0$, and we write $\mathcal{L}$ for the resulting line bundle with connection on G_N. The basic remark of Deligne is

Proposition 3.2. *Assume*

$$H^0_{DR}(C-D,(L,\nabla)) = H^2_{DR}(C-D,(L,\nabla)) = 0.$$

Then

$$\begin{aligned}\det(H^*_{DR}(C-D,(L,\nabla)) &= \det(H^1_{DR}((C-D,(L,\nabla)))) \\ &\cong H^N(Sym^N(C-D),(\pi^*(\mathcal{L},\nabla)))\end{aligned} \tag{3.4}$$

as a line with connection on K.

Proof. Our hypotheses imply $H^1_{DR}(C-D,(L,\nabla))$ has dimension N. Consider the diagram

$$\begin{array}{ccc} (C-D)^N & \xrightarrow{p} & \mathrm{Sym}^N(C-D) \\ & \searrow q & \downarrow \pi \\ & & G_N \end{array} \tag{3.5}$$

We have $q^*(\mathcal{L}) \cong L \boxtimes \cdots \boxtimes L$ (exterior tensor product on $(C-D)^N$. The Künneth formula gives

$$H^N((C-D)^N,(L\boxtimes\cdots\boxtimes L,\nabla)) \cong H^1_{DR}(C-D,(L,\nabla))^{\otimes N}. \tag{3.6}$$

There is an action of the symmetric group $\mathcal{S}_N$ on the pair

$$((C-D)^N, L\boxtimes\cdots\boxtimes L).$$

The resulting action on $(H^1_{DR})^{\otimes N}$ is alternating because of the odd degree cohomology, so the invariants are precisely $\det H^1_{DR}$. There is

an evident map

(3.7)
$$p^* : H^N_{DR}(\mathrm{Sym}^N(C-D), \pi^*\mathcal{L}) \to H^N_{DR}((C-D)^N, L \boxtimes \cdots \boxtimes L)^{\mathcal{S}_N} \\ = \det H^1_{DR}(C-D, L)$$

To show this map is an isomorphism, it suffices to remark that one has a trace map

$$(3.8)\quad p_* : H^N_{DR}((C-D)^N, L \boxtimes \cdots \boxtimes L) \to H^N_{DR}(\mathrm{Sym}^N(C-D), \pi^*\mathcal{L})$$

Because $L \boxtimes \cdots \boxtimes L = p^*\pi^*\mathcal{L}$, the existence of such a trace follows from the projection formula and the trace in de Rham cohomology with constant coefficients. □

Now one uses the geometry of the map π and (3.4) to compute the determinant.

Lemma 3.3. *Let X be a smooth variety over a field of characteristic 0. Let $A \subset X$ be a smooth subvariety of codimension p. Let (E, ∇) be an integrable connection on X. Then*

$$(3.9)\qquad \mathbb{H}^n_A(X, E \otimes \Omega^*_X) \cong H^{n-2p}(A, E \otimes \Omega^*_A).$$

Proof. Write $\underline{\underline{H}}^r_A(F)$ for the Zariski sheaf associated to the presheaf $U \mapsto H^r_A(U, F)$ for any Zariski sheaf F on X. For F locally free, $\underline{\underline{H}}^r_A(F) = (0)$ for $r \neq p$ by purity. Duality theory gives (here $X \supset A_\alpha \supset A$ runs through nilpotent thickenings)

$$(3.10)\quad E \otimes \Omega^m_A \to \mathrm{Ext}^p(\mathcal{O}_A, E \otimes \Omega^{m+p}_X) \to \varinjlim{}_\alpha \mathrm{Ext}^p(\mathcal{O}_{A_\alpha}, E \otimes \Omega^{m+p}_X) \\ \cong \underline{\underline{H}}^p_A(E \otimes \Omega^{m+p}_X).$$

We want to show that this map is an isomorphism, compatible with the connection, thus yielding a quasiisomorphism of complexes

$$(3.11)\qquad E \otimes \Omega^*_A \to \underline{\underline{H}}^p_A(E \otimes \Omega^*_X).$$

The problem is local, so we can assume

$$A \subset A_1 \subset \ldots \subset A_p = X$$

with A_i smooth of codimension $p-i$ in X. Now $\underline{\underline{H}}^p_A(E \otimes \Omega^*_X)$ represents $R\underline{\underline{\Gamma}}_A(E \otimes \Omega^*_X)[p]$ in the derived category of Zariski sheaves on A, and in the derived category we may write

(3.12)
$$R\underline{\underline{\Gamma}}_A(E \otimes \Omega^*_X)[p] = R\underline{\underline{\Gamma}}_A[1] \circ R\underline{\underline{\Gamma}}_{A_1}[1] \circ \ldots \circ R\underline{\underline{\Gamma}}_{A_{p-1}}(E \otimes \Omega^*_X)[1]$$

In this way, we reduce to verifying (3.9) in the case $p = 1$. So, suppose $A : t = 0$ in $X = \mathrm{Spec}\,(R)$. We have

$$\underline{\underline{H}}^1_A(E \otimes \Omega^*_X) \cong E_{R[t^{-1}]} \otimes \Omega^*_{R[t^{-1}]} / E_R \otimes \Omega^*_R \tag{3.13}$$

as $H^1_{DR}(X, E) \subset H^1_{DR}(X - A, E)$.

By [4], since E has no singularity along $t = 0$, one has

$$(E_R \otimes \Omega^*_R(\log(t = 0)), \nabla) \overset{\text{q.iso.}}{\to} (E_{R[t^{-1}]} \otimes \Omega^*_{R[t^{-1}]}, \nabla|_{\mathrm{Spec} R[t^{-1}]}) \tag{3.14}$$

Thus res $: \underline{\underline{H}}^1_A(E \otimes \Omega^*_X, \nabla)) \to E_{R/tR} \otimes (\Omega^*_{R/tR}[-1], \nabla|_{\mathrm{Spec}(R/tR)})$ is an isomorphism. $\square$

Lemma 3.4. *Let $p : G_N \to J_N$ be the projection to the corresponding torseur over the absolute jacobian. Write $[\Omega^1_{C/K}(\mathcal{D})] \in J_N$ for the point corresponding to the canonical bundle twisted by $\mathcal{O}(\mathcal{D})$. Let $a \in G_N$. We have*

$$\pi^{-1}(a) = \begin{cases} \mathbb{A}^{g-1} & p(a) \neq [\Omega^1_{C/K}(\mathcal{D})] \\ \mathbb{A}^g & p(a) = [\Omega^1_{C/K}(\mathcal{D})];\ \partial(a) = 0 \\ \emptyset & p(a) = [\Omega^1_{C/K}(\mathcal{D})];\ \partial(a) \neq 0 \end{cases} \tag{3.15}$$

where by definition $\mathbb{A}^{g-1} = \emptyset$ is $g = 0$. Note that if $p(a) = [\Omega^1_{C/K}(\mathcal{D})]$, then a corresponds to a trivialization $\mathcal{O}_\mathcal{D} \cong \Omega^1_{C/K}(\mathcal{D})|_\mathcal{D}$ defined upto scalar. $\partial(a)$ in the above refers to the evident boundary of this trivialization in $H^1(\Omega^1_{C/K}) = K$ (again upto scale).

Proof. Let $p(a)$ correspond to a line bundle M of degree N, we consider the exact sequence

$$0 \to M(-\mathcal{D}) \to M \to M|_\mathcal{D} \to 0 \tag{3.16}$$

Suppose first $M \neq \Omega^1_{C/K}(\mathcal{D})$. Then $H^1(M(-\mathcal{D})) = (0)$, so any trivialization in $H^0(M|_\mathcal{D})$ lifts to $H^0(M)$, and the space of such liftings is a torseur under $H^0(M(-\mathcal{D}))$, a vector space of dimension $g - 1$. (Note this is an affine torseur, not a projective torseur.) If $M = \Omega^1_{C/K}(\mathcal{D})$, $H^0(M(-\mathcal{D}))$ has dimension g, and the image $H^0(M) \to H^0(M|_\mathcal{D})$ has codimension 1. $\square$

Remark 3.5. *If we choose local parameters t_i at $c_i \in \mathcal{D}$, then*

$$H^0(\Omega^1_{C/K}(\mathcal{D})|_\mathcal{D})$$

can be identified with the space of polar parts of 1*-forms with poles along* $\mathcal{D}$*, and the map* ∂ *is given by the residue*

$$\partial(\sum_i \sum_{j=0}^{m_i-1} u_{ij}dt_i/t_i^{m_i-j}) = \sum_i u_{i,m_i-1} \tag{3.17}$$

Note the (open) condition for an element in $H^0(\Omega^1_{C/K}(\mathcal{D})|_{\mathcal{D}})$ *to be a trivialization is simply*

$$\prod_i u_{i0} \neq 0. \tag{3.18}$$

Because $B \subset G_N$*, we must factor out by the action of* $\mathbb{G}_m$*, which we can normalize away by setting* $u_{10} = 1$*. Thus we have*

$$B := \pi(Sym^N(C-D)) \cap p^{-1}[\Omega^1_{C/K}(\mathcal{D})] = \tag{3.19}$$
$$\Big\{ \sum_i \sum_{j=0}^{m_i-1} u_{ij}dt_i/t_i^{m_i-j} \;\Big|\; \sum_i u_{im_i-1} = 0;\; \prod_i u_{i0} \neq 0;\; u_{10} = 1 \Big\}.$$

Define $A := \pi^{-1}(B) \subset \mathrm{Sym}^N(C-D) =: X$. Using the localization sequence and lemma 3.4, we get an exact sequence

(3.20)
$$\begin{array}{ccccc} H^{N-2g}_{DR}(A/K, \pi^*\mathcal{L}) & \longrightarrow & H^N_{DR}(X/K, \pi^*\mathcal{L}) & \longrightarrow & H^N_{DR}(X-A/K, \pi^*\mathcal{L}) \\ \uparrow\cong & & & & \uparrow\cong \\ H^{N-2g}_{DR}(B/K, \mathcal{L}|_B) & & & & H^N_{DR}(G_N - p^{-1}[\Omega^1_{C/K}(\mathcal{D})]/K, \mathcal{L}) \end{array}$$

where $p : G_N \to J_N$ with $J = J(C)$ the absolute Jacobian. The fact that the vertical arrows in this diagram are isomorphisms follows because the maps are maps of affine space bundles and the line bundles with connection are pulled back from the base.

To simplify the presentation, we will assume that $m_1 \geq 2$. Another proof of our formula for the de Rham determinant in the case $\mathcal{D} = D = \sum c_i$, (i.e. for regular singular points) will be given in theorem 4.6.

Lemma 3.6. *Assume the line bundle* L *on* C *has degree* 0*, and that* $\mathcal{D}$ *is minimal, i.e.* $\nabla : L|_{\mathcal{D}} \cong L \otimes \Omega^1_{C/K}(\mathcal{D})|_{\mathcal{D}}$*. We continue to assume also that* $\mathcal{D} = \sum m_i c_i$ *with* $m_1 \geq 2$*. Then*

$$H^*_{DR}(Sym^N(C-D) - A/K, \pi^*\mathcal{L}) = (0). \tag{3.21}$$

Proof. The isomorphism on the right in (3.20) implies we must show $H^*_{DR}(G_N - p^{-1}[\Omega^1_{C/K}(\mathcal{D})]/K, \mathcal{L}) = (0)$. The assumption $m_1 \geq 2$ means we have a $\mathbb{G}_a$ action by translation on $G_N - p^{-1}[\Omega^1_{C/K}(\mathcal{D})]/K, \mathcal{L}$, and minimality of $\mathcal{D}$ implies that the connection is nontrivial on the fibres.

The fibration is Zariski-locally trivial, so the Leray spectral sequence for de Rham cohomology reduces us to showing $H^*_{DR}(\mathbb{G}_{a,S}/S, (\mathcal{O}, \Xi)) = (0)$ where Ξ is an everywhere non-zero, translation invariant, relative 1-form on $\mathcal{O}_{\mathbb{G}_{a,S}}$. In other words, for $s \in \mathcal{O}_S^\times$ we must show

$$\mathcal{O}_S[t] \xrightarrow{d+sdt} \mathcal{O}_S[t]dt \tag{3.22}$$

has trivial cohomology. This is straightforward. □

Lemma 3.7. *Assume $H^0_{DR}(C-D, L) = (0)$. Then*

$$H^{m-2}_{DR}(B/K, \mathcal{L}|_B) \cong H^N_{DR}(Sym^N(C-D)/K, \pi^*\mathcal{L}) \cong K \tag{3.23}$$

as a line with a connection over K.

Proof. Note $m-2 = N-2g$. Extending the top sequence in (3.20) one step to the left and using the previous lemma gives the left isomorphism. We have already seen the isomorphism on the right. □

Our task now is to calculate $H^{m-2}_{DR}(B/K, \mathcal{L}|_B)$ with its connection. We assume that $\mathcal{L} \in \mathrm{Pic}^0(J)$ as in lemma 3.6. Then $\mathcal{L}$ carries a relative invariant connection $d_{/K}$ on J, and $\nabla_{/K} = d_{/K} + \Xi$ for some invariant form $\Xi \in H^0(G, \Omega^1_{G/K})^{\mathrm{inv}}$. Changing the choice of $d_{/K}$ changes Ξ to $\Xi + p^*(\alpha)$, where $\alpha \in H^0(J, \Omega^1_{J/K})^{\mathrm{inv}}$ and $p : G_N \to J_N$ is the torseur under the affine group $\mathcal{G} := \ker(G \to J)$. In particular, $\Xi|p^{-1}[\Omega^1_{C/K}(\mathcal{D})]$ does not depend on the choice of $d_{/K}$. As $p^{-1}(\Omega^1_{G/K}(\mathcal{D}))$ is isomorphic to $\mathcal{G}$, and we see that

$$(\mathcal{L}, \nabla_{/K})|_B = (\mathcal{O}_B, d + \Xi|_B). \tag{3.24}$$

We say that $\Xi|_B$ vanishes at a point $b \in B$ if $\Xi(b) = 0$ in $\mathfrak{m}_b/\mathfrak{m}_b^2$.

Lemma 3.8. *Let $b \in B \subset G_N$ correspond to the trivialization on $\Omega^1_{C/K}(\mathcal{D})|_{\mathcal{D}}$ given by $\nabla|_{\mathcal{D}} : L|_{\mathcal{D}} \to L \otimes \Omega^1_{C/K}(\mathcal{D})|_{\mathcal{D}}$. Then $\Xi|_B$ vanishes at b. $\Xi|_B$ does not vanish at any other point of B.*

Proof. Let us write $\eta := i^*\Xi|C-D$, where $i : C-D \to G$ is the cycle map. Then by definition, the trivialization of $\Omega^1_{C/K}(\mathcal{D})|_{\mathcal{D}}$ associated to $i^*(\mathcal{L}, \nabla)$ depends only on $\eta|\mathcal{D}$, or equivalently only on $\Xi|p^{-1}[\Omega^1_{C/K}(\mathcal{D})]$. We have

$$\pi^*\Xi = \sum_{i=1}^N \eta_i \tag{3.25}$$

where η_i is the pullback of η via the i-th projection $(C-D)^N \to C-D$.

Suppose for a moment that the divisor of η (viewed as a section of $\Omega^1_{C/K}(\mathcal{D})$) is reduced, $(\eta) = \sum e_i$; $e_i \in (C-D)(\bar{K})$. Let $e :=$

$(e_1, \cdots, e_N) \in \mathrm{Sym}^N(C-D)$ be the point corresponding to η. We have $e \in A = \pi^{-1}(B) \subset \mathrm{Sym}^N(C-D)$ and $b = \pi(e)$. Since $A \to B$ is a projective bundle, there is a surjection on tangent spaces

$$\begin{array}{ccc} T_A(e) & \hookrightarrow & T_{\mathrm{Sym}^N(C-D)}(e) \\ \text{surjective} \downarrow & & \downarrow \pi_* \\ T_B(b) & \hookrightarrow & T_{G_N}(b) \end{array} \tag{3.26}$$

Since $T_{\mathrm{Sym}^N(C-D)}(e)$ is spanned by expressions $\sum \tau_i|_{e_i}$, to show $\Xi|_B$ vanishes at b, it suffices to show

$$< \Xi, \pi_*(\sum \tau_i|_{e_i}) >= 0. \tag{3.27}$$

This expression equals

$$< \pi^*(\Xi), \sum \tau_i|_{e_i} >= \sum < \eta, \tau_i|_{e_i} > . \tag{3.28}$$

Each term on the right vanishes because $\eta(e_i) = 0$ in $\mathfrak{m}_{e_i}/\mathfrak{m}_{e_i}^2$. The general case ($(\eta)$ not necessarily reduced) follows from this by a specialization argument.

We postpone until lemma 3.10 the proof that $\Xi|_B$ doesn't vanish at any other point of B. $\square$

By assumption we start with an absolute, invariant, integrable connection on $\mathcal{L}$ of degree 0 on J. Restricting to B, we get an absolute closed invariant 1-form Ψ, whose corresponding relative form is Ξ. Recall (3.19) we have coordinates u_{ij} on B with $u_{10} = 1$, $\prod u_{i0} \neq 0$, and $u_{1,m_1-1} = -\sum_{i\geq 2} u_{i,m_i-1}$.

Lemma 3.9. *Under the assumption that*

$$\{i^*\nabla_{/K} : i^*\mathcal{L} \to i^*\mathcal{L}\otimes\Omega^1_{C/K}(\mathcal{D})\} \to \{j_*i^*\mathcal{L}|(C-D) \to j_*i^*\mathcal{L}\otimes\Omega^1_{(C-D)/K}\}$$

is a quasiisomorphism, and $i^*\nabla_{/K}$ has poles along all points of D, we can arrange that a K-basis for

$$H^{m-2}_{DR}(B/K, \mathcal{L}|_B) = H^{m-2}_{DR}(B/K, (\mathcal{O}_B, d + \Xi))$$

is given by the closed form

$$\theta := \prod_{\substack{(i,j) \\ m_i \geq 2}} du_{ij} \wedge \prod_{\substack{i \\ m_i = 1}} \frac{du_{i0}}{u_{i0}} \tag{3.29}$$

Proof. Recall (2.27) the relative invariant forms on $\mathcal{G} := \ker(G \to J)$ are the τ_{ij} defined by the expression

$$\sum_{j=0}^{m_i-1} \tau_{ij} T_i^j = \Big(\sum_j u_{ij} T_i^j\Big)^{-1} \sum_j du_{ij} T_i^j. \tag{3.30}$$

Write

$$\Xi = \sum_{i,j} \lambda_{ij}\tau_{ij}; \quad \lambda_{i,m_i-1} \neq 0 \tag{3.31}$$

where the nonvanishing condition comes from the requirement that the form restricted to $C - D$ gives a trivialization along $\mathcal{D}$ (see (2)). If we write $(\mod T_i^{m_i})$

(3.32)

$$\Big(u_{i0} + u_{i1}T_i + \ldots + u_{i,m_i-1}T_i^{m_i-1}\Big)^{-1} = \nu_{i0} + \nu_{i1}T_i + \ldots + \nu_{i,m_i-1}T_i^{m_i-1}$$

we get the table

$$\begin{aligned} \tau_{i0} &= \nu_{i0}du_{i0} \\ \tau_{i1} &= \nu_{i0}du_{i1} + \nu_{i1}du_{i0} \\ &\vdots \\ \tau_{i,m_i-1} &= \nu_{i0}du_{i,m_i-1} + \ldots + \nu_{i,m_i-1}du_{i0} \end{aligned} \tag{3.33}$$

Note if we give $u_{ij},\ du_{ij},\ \nu_{ij}$ all weight j, then τ_{ij} will be homogeneous of weight j. Comparing (3.33) and (3.31), it follows that if we expand $\Xi|_B$ in terms of the du_{ij}, omitting du_{10} and du_{1,m_1-1}, we find for suitable $\alpha_{ij} \neq 0$

$$\Xi|_B = \sum_i g_{ij}du_{ij} = \sum_{i,j} \Big[(\alpha_{i,m_i-1}u_{i0}^{-1} - \alpha_{1,m_1-1})du_{i,m_i-1} + \tag{3.34}$$

$$u_{i0}^{-1}\sum_{p=0}^{m_i-2}\Big(\alpha_{ip}\frac{u_{i,m_i-1-p}}{u_{i0}} + \sum \text{ terms at least quadratic in } \frac{u_{ik}}{u_{i0}}\Big)du_{ip}\Big]$$

Looking at the weights, we see that for $p \leq m_i - 2$

$$g_{ip} = \text{ nonzero multiple of } \frac{u_{i,m_i-p-1}}{u_{i0}} + \\ \text{terms only involving } u_{ij},\ j < m_i - p - 1 \tag{3.35}$$

while

$$g_{i,m_i-1} = \alpha_{i,m_i-1}u_{i0}^{-1} - \alpha_{1,m_1-1} \tag{3.36}$$

with neither α coefficient 0.

Now generators of $H^{m-2}_{DR}(B, (\mathcal{O}, d + \Xi))$ are of the form $M\theta$ where M is a monomial in the u_{ij}, u_{i0}^{-1} and θ is as in (3.29). Relations are

$$\Big(\frac{\partial}{\partial u_{ij}} + g_{ij}\Big)(M)\theta = 0 \tag{3.37}$$

Because of (3.35) one can use these relations to eliminate $u_{ij},\ j > 0$ from M by downward induction on j, starting from u_{i,m_i-1}. We are left with the case $M = u_{2,0}^{n_2} \cdots u_{r,0}^{n_r}$ with $n_i \in \mathbb{Z}$. In this case we can apply (3.36). If $m_i \geq 2$, we get the relation

$$u_{2,0}^{n_2} \cdots u_{r,0}^{n_r}\theta \equiv \frac{\alpha_{i,m_i-1}}{\alpha_{1,m_1-1}} u_{2,0}^{n_2} \cdots u_{i,0}^{n_i-1} \cdots u_{r,0}^{n_r}\theta \tag{3.38}$$

Using this, we can get $n_i = 0$. If $m_i = 1$ and $i \geq 2$ the relation becomes

$$u_{2,0}^{n_2} \cdots u_{r,0}^{n_r}\theta \equiv \frac{n_i + \alpha_{i,0}}{\alpha_{1,m_1-1}} u_{2,0}^{n_2} \cdots u_{i,0}^{n_i-1} \cdots u_{r,0}^{n_r}\theta \tag{3.39}$$

If $\alpha_{i,0}$ is not a positive integer, we can arrange $n_i = 0$.

On the other hand, we claim that if $m_i = 1$, then α_{i0} is the residue of $i^*\nabla$ along c_i. Indeed (2.27) shows that in this case, $\tau_{i0} = d\log u_{i0}$, and u_{i0} is then just the local parameter in the point c_i (see (2.26)). Thus the quasiisomorphism

$$\begin{aligned} \{i^*\nabla_{/K} : i^*\mathcal{L} \to i^*\mathcal{L} \otimes \Omega^1_{C/K}(\mathcal{D})\} \to \{j_*i^*\mathcal{L}|(C-D) \\ \to j_*i^*\mathcal{L} \otimes \Omega^1_{(C-D)/K}\} \end{aligned} \tag{3.40}$$

forces a_i not to lie in $\mathbb{N} - \{0\}$. □

The following was left open in the proof of lemma 3.8:

Lemma 3.10. *Let $\Xi = \sum_{i,j} \lambda_{ij}\tau_{ij}$ be as in lemma 3.8. Then Ξ vanishes at a unique point $b \in B$.*

Proof. We have seen in the proof of lemma 3.8 that Ξ vanishes at a point in B. We must show it vanishes at at most one point. Let $b = (\ldots, b_{ij}, \ldots) \in B$ be a point. Write $b = (\ldots, y_{ij}, \ldots)$ with respect to the coordinates ν_{ij} (3.32). Staring at (3.33), the conditions that $\Xi|_b = 0$ are seen to be (recall $\sum_i du_{i,m_i-1} = 0 = du_{10}$) for $i \geq 2$

$$\begin{gathered} \lambda_{i,m_i-1}y_{i1} + \lambda_{i,m_i-2}y_{10} = 0 \\ \lambda_{i,m_i-1}y_{i2} + \lambda_{i,m_i-2}y_{i1} + \lambda_{i,m_i-3}y_{i0} = 0 \\ \vdots \ \vdots \\ \lambda_{i,m_i-1}y_{i,m_i-1} + \lambda_{i,m_i-2}y_{i,m_i-2} + \ldots + \lambda_{i,0}y_{i0} = 0 \end{gathered} \tag{3.41}$$

For $i = 1$ one gets the same list but with the last line (coefficient of du_{i0}) omitted. Finally, using $\nu_{10} = 1$ and $du_{1,m_1-1} = -\sum_{i\geq 2} du_{i,m_i-1}$ one gets

$$\lambda_{1,m_1-1} = y_{i0}\lambda_{i,m_i-1}; \quad 2 \leq i \leq r \tag{3.42}$$

Since $\lambda_{i,m_i-1} \neq 0$, equations (3.41) and (3.42) admit a unique solution for the y_{ij}. Since we know Ξ vanishes at at least one point of B, this point must lie in B. □

Finally, we must calculate the Gauß-Manin connection on

$$H^{m-2}_{DR}(B, (\mathcal{O}_B, d + \Xi))$$

Define Ψ to be an absolute invariant form lifting Ξ. By assumption our connection on $C - D$ comes from an absolute integrable connection which, by proposition 2.17, comes from an absolute integrable connection on G. Restricting this connection to B gives our Ψ.

Lemma 3.11. *With notation as above, there exists $F \in \mathcal{O}_B$, $\eta \in \Omega^1_K$, and $a_i \in k$, $i \geq 2$, such that*

$$\Psi = \sum_{i=2}^{r} a_i \frac{du_{i0}}{u_{i0}} + dF + \eta. \tag{3.43}$$

If moreover ∇ is integrable, then η is closed.

Proof. Since Ξ is (relatively) closed on B, one can write

$$\Xi = \sum_{i \geq 2} a_i \frac{du_{i0}}{u_{i0}} + d_{/K} F; \quad a_i \in K. \tag{3.44}$$

Lifting to an absolute form forces

$$\Psi = \sum_{i \geq 2} a_i \frac{du_{i0}}{u_{i0}} + dF + \sum_j f_j \eta_j; \quad f_j \in \mathcal{O}_B,\ \eta_j \in \Omega^1_K. \tag{3.45}$$

Here the η_j are linearly independent in Ω^1_K. Using $d\Psi = 0$ modulo $\Omega^2_K \otimes \mathcal{O}_B$ and taking residues along $u_{i0} = 0$ yields $a_i \in k \subset K$. Then computing $d\Psi \mod \mathcal{O}_B \otimes \Omega^2_K$ yields

$$0 = \sum_j d_{/K} f_j \otimes \eta_j \in \Omega^1_{B/K} \otimes \Omega^1_K. \tag{3.46}$$

It follows that $f_j \in K$, so $\eta := \sum f_j \eta_j \in \Omega^1_K$. Taking d again shows η is closed if ∇ is integrable. $\square$

We now compute the Gauß-Manin connection. We have the diagram of global sections

$$\begin{tikzcd} & \Omega^{m-2}_{B/k} \arrow[r, "\text{onto}"'] \arrow[d, "d+\Psi"] & \Omega^{m-2}_{B/K} \\ \Omega^{m-2}_{B/K} \wedge \Omega^1_K \arrow[r, "\cong"'] & \frac{\Omega^{m-1}_{B/k}}{\Omega^{m-3}_{B/k} \cdot \Omega^2_K} & \end{tikzcd} \tag{3.47}$$

The connection is determined by its value on θ (3.29). To calculate, one lifts θ to $\tilde{\theta} \in \Omega^{m-2}_{B/k}$ and then applies $d + \Psi$. But for $\tilde{\theta}$ one can

choose the form with the same expression (3.29). This form is closed, so

$$\nabla_{GM}(\theta) = \Psi \wedge \theta = (d_K(F) + \eta) \wedge \theta \tag{3.48}$$

Here we write $F = \sum_I a_I u^I,\ a_i \in K$ and $d_K(F) := \sum da_I u^I$.

Let $b \in B$ be the point corresponding to the trivialization of $\Omega^1_{C/K}(\mathcal{D})$ given by the polar part of the original relative connection. It really lies in B since we have assumed that $\deg \mathcal{L} = 0$.

Lemma 3.12. *With notation as above, the Gauß-Manin connection on the rank* 1 *K-vector space*

$$H^{m-2}_{DR}(B, (\mathcal{O}, d + \Xi))$$

described by (3.48) *is isomorphic to the connection on K given by*

$$1 \mapsto \Psi|_b + \frac{1}{2} d\log(\kappa)$$

for a suitable $\kappa \in K^\times$.

Proof. We have seen (lemma 3.8) that this point b is determined by the condition that $\Xi(b) = 0 \in \mathfrak{m}_{B,b}/\mathfrak{m}^2_{B,b}$. Changing Ψ by a closed form pulled back from K changes the Gauß-Manin connection and the connection at b in the same way, so we can assume $\eta = 0$, i.e. $\Psi = \sum a_i \frac{du_{i0}}{u_{i0}} + dF$. Write

$$g_{ij} = \begin{cases} \frac{\partial F}{\partial u_{ij}} & j > 0 \\ \frac{\partial F}{\partial u_{i0}} + a_i/u_{i0} & j = 0. \end{cases} \tag{3.49}$$

Write $F = \sum_I a_I u^I$. Then

$$\Psi = \sum g_{ij} du_{ij} + \sum_I u^I da_I; \quad \Psi \wedge \theta = \sum_I u^I da_I \wedge \theta \tag{3.50}$$

We have

$$g_{ij}(b) = 0,\ j < m_i - 1; \quad g_{i,m_i-1}(b) = g_{k,m_k-1}(b);\ \text{all } i, k. \tag{3.51}$$

Since $\sum_i du_{i,m_i-1}|_B = 0$, we see from (3.51) that

$$\Psi|_{\{b\}} = \sum_I b^I da_I \tag{3.52}$$

Thus, it will suffice to relate $u^I\theta$ and $b^I\theta$ in H^{m-2}_{DR}. Note that each monomial u^I involves u_{ij} for only one value of i, and the weight of u^I is $\le m_i - 1$ (see the discussion after (3.33)).

Suppose first the weight of u^I is strictly less than $m_i - 1$. Let j be maximal such that u_{ij} appears in u^I. From (3.35) it follows that

$$g_{i,m_i-1-j} = \alpha_{i,m_i-1-j}\frac{u_{ij}}{u_{i0}^2} + \text{terms involving only } u_{ik};\ k < j. \tag{3.53}$$

Here $\alpha_{i,m_i-1-j} \neq 0$. Define $u^L = u^I u_{i0}^2 u_{ij}^{-1}$. Note the weight of u^L is $< m_i - 1 - j$, so in H^{m-2}_{DR} we have (compare (3.37))

$$\begin{aligned} u^I\theta = (u^I - \alpha^{-1}_{i,m_i-1-j}(\frac{\partial}{\partial u_{i,m_i-1-j}} + g_{i,m_i-1-j})u^L)\theta = \\ (u^I - \alpha^{-1}_{i,m_i-1-j}g_{i,m_i-1-j}u^L)\theta = \sigma_I\theta \end{aligned} \tag{3.54}$$

where σ_I is a sum of terms of weights $< |I|$ and terms of weight $|I|$ only involving $u_{i0}, \ldots, u_{i,j-1}$. Note that $b^I = \sigma_I(b)$ because $g_{i,m_i-1-j}(b) = 0$. In this way we reduce to the case $u^I = u_{i0}^p$. Our assumption on the weight implies $m_i \geq 2$, so

$$(\frac{\partial}{\partial u_{i,m_i-1}} + g_{i,m_i-1})u_{i0}^p = g_{i,m_i-1}u_{i0}^p. \tag{3.55}$$

Together with (3.35) and $g_{ij}(b) = 0$, this enables us to reduce to $p = 0$.

Suppose now the weight of I is $m_i - 1$. If $m_i \geq 2$ we can use the above argument, except in the case $u^I = u_{ij}u_{i,m_i-1-j}u_{i0}^{-2}$. Here there are two subcases. If $j \neq m_i - 1 - j$, the α_{i,m_i-1-j} in (3.54) is a_I in the expansion $F = \sum_I a_I u^I$, so

$$u^I da_I \wedge \theta = (b^I da_I + \frac{da_I}{a_I}) \wedge \theta. \tag{3.56}$$

This completes the proof in this case because the connections $b^I da_i$ and $b^I da_I + d\log(a_I)$ are isomorphic. If, on the other hand, m_i is odd and $j = \frac{m_i-1}{2}$, the monomial $u^I = u_{ij}^2 u_{i0}^{-2}$ and dF contains the term $2a_I u_{ij} du_{ij}$. Thus, from (3.53) we conclude $\alpha_{i,\frac{m_i-1}{2}} = 2a_I$. The lefthand identity in (3.54) yields in this case

$$u^I da_I \wedge \theta = (b^I da_I + \frac{1}{2}d\log a_I) \wedge \theta. \tag{3.57}$$

In the statement of the lemma, we take κ to be the product of the corresponding a_I.

Suppose finally $m_i = 1$. In this case $\frac{\partial F}{\partial u_{i0}} = 0$, so the corresponding $a_I = 0$ and by (3.52) this term contributes nothing to $\Psi|_{\{b\}}$. Similarly, by (3.50) there is no contribution to $\Psi \wedge \theta$.

$\square$

We give two interpretations of the 2-torsion term $\frac{1}{2}d\log(\kappa)$ occurring in the previous lemma.

Definition 3.13. *Let σ be a closed 1-form relative to K on*

$$\mathrm{Spec}\,(K[[t_1,\dots,t_N]]).$$

Assume $\sigma(0) = 0 \in \mathfrak{m}/\mathfrak{m}^2$. Write $\sigma = dh$ with $h(0) = 0$, so $h = h_2 + h_3 + \dots$ with h_i homogeneous of degree i. If h_2 is nondegenerate, we may define disc(σ) = discriminant(h_2) $\in K^\times/K^{\times 2}$. This is well-defined independent of the choice of parameters.

Theorem 3.14. *The Gauß-Manin connection on $H^{m-2}_{DR/K}(B,(\mathcal{O}, d+\Xi))$ is isomorphic to*

$$(d+\Psi)|_{\{b\}} + \frac{1}{2} d\log(\mathrm{disc}(\Xi|_{\widehat{\mathcal{O}}_{B,b}})).$$

(In particular, the quadratic term in $h = \int \Xi|_{\widehat{\mathcal{O}}_{B,b}}$ is non-degenerate.)

Proof. First we collect some facts about $\Xi = \sum_{i,j}\lambda_{ij}\tau_{ij} = \sum g_{ij}du_{ij}$. We have $u_{1,0} = 1$ and $\nu_{i,0} = u_{i,0}^{-1}$. It follows from (3.33) of the paper that u_{1,m_1-1} does not appear in the expression for Ξ and du_{1,m_1-1} only appears with constant coefficient. Restricting to $B : u_{1,m_1-1} = -\sum_{i\geq 2} u_{i,m_i-1}$ thus has the effect of surpressing the term in du_{1,m_1-1} and changing the coefficients g_{i,m_i-1} by a constant for $i \geq 2$. Expressed in this way, it follows that the coefficient of du_{ij} in $\Xi|_B$ involves only monomials in u_{ip} for the same i. Giving u_{ij} and du_{ij} both weight j, the terms in $g_{ij}du_{ij}$ all have weight $\leq m_i - 1$. It follows from formulas (3.34)-(3.36) that, writing $U_{ij} = u_{ij} - b_{ij}$ so $U_{ij}(b) = 0$, we may write

$$\Xi|_B = \sum G_{ij}(U)dU_{ij}.$$

Here $G_{ij}(0) = 0$. giving U_{ij} and dU_{ij} weights j, all terms with first index i have weights $\leq m_i - 1$. All terms of the form

$$U_{ij}dU_{i,m_i-1-j},\quad 0 \leq j \leq m_i - 1,\ i \geq 2 \quad (\text{resp. } i = 1,\ 1 \leq j \leq m_1 - 2)$$

occur with nonzero coefficient. Notice that replacing u_{ij} with $U_{ij} + b_{ij}$ introduces monomials of lower degree, but these have weight $< m_i - 1$.

It follows that $\mathrm{disc}(\Xi|_{\widehat{\mathcal{O}}_{B,b}})$ is the determinant of a matrix

$$M = \begin{pmatrix} M_1 & 0 & \dots & \dots & 0 \\ 0 & M_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & M_r \end{pmatrix}$$

where M_i is symmetric, $m_i \times m_i$ (resp. $(m_1 - 2) \times (m_1 - 2)$), and has the shape

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \bullet \\ \dots & \dots & \dots & \bullet & 0 \\ \dots & \dots & \bullet & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bullet & 0 & \dots & \dots & 0 \end{pmatrix}$$

with the entries $\bullet$ non-zero.

Mod squares, $\det(M_i)$ is 1 if m_i is even, and is given by

$$\frac{1}{2} \cdot \text{coefficient of}(U_{i,\frac{m_i-1}{2}} dU_{i,\frac{m_i-1}{2}})$$

if m_i is odd. Writing

$$\Psi|_B = \sum a_i \frac{du_{i0}}{u_{i0}} + dF \tag{3.58}$$

as just above (3.49) with $F = \sum a_I u^I$ we find

$$\frac{1}{2} d \log \operatorname{disc}(\Xi|_{\widehat{\mathcal{O}}_{B,b}}) = \frac{1}{2} \sum \frac{da_I}{a_I} = \frac{1}{2} d \log(\kappa).$$

(The sum on the right is over all I such that $u^I = (u_{i,\frac{m_i-1}{2}})^2$.) $\square$

Another interpretation of the 2-torsion is the following. As in (2.25) for s_i a local parameter at $c_i \in \mathcal{D}$ and t_i another copy of s_i (so $s_i - t_i$ is a local defining equation for the diagonal in $(C \times C)$, the pullback of u_{ip} to $K((s_i))$ is the coefficient $s_i^{-(p+1)}$ of t_i^p in $(s_i - t_i)^{-1}$. It follows from (3.30) that

$$\tau_{ij} \text{ pulls back to } \frac{-ds_i}{s_i^{j+1}}. \tag{3.59}$$

Write the polar part of the connection at $s_i = 0$ in the form $(g_0 + g_1 s_i + \dots)\frac{ds_i}{s_i^{m_i}}$. Since $\Xi = \sum_{i,j} \lambda_{ij} \tau_{ij}$ pulls back to this connection form, we get $g_0 = -\lambda_{i,m_i-1}$. On the other hand, again from (3.30) the coefficient of $u_{i,0}^{-2} u_{i,\frac{m_i-1}{2}} du_{i,\frac{m_i-1}{2}}$ in Ξ is $-\lambda_{i,m_i-1}$ if m_i is odd. This coefficient is the contribution to $\operatorname{disc}(\Xi|_{\mathcal{O}_{B,b}})$ from the point $c_i \in \mathcal{D}$, so we conclude

Theorem 3.15. *Write the relative connection at $c_i \in \mathcal{D}$ in the form $(g_{i,0} + g_{i,1} s_i + \dots)\frac{ds_i}{s_i^{m_i}}$. Then the Gauß-Manin connection on $H^{m-2}_{DR/K}(B, (\mathcal{O}, d + \Xi))$ is isomorphic to*

$$(d + \Psi)|_{\{b\}} + \sum_i \frac{m_i}{2} d \log(g_{i,0}(0)).$$

Definition 3.16. *With notation as above, write*

$$\tau(L) = \sum_i \frac{m_i}{2} d\log(g_{i,0}(0)).$$

To summarize, we have proven

Theorem 3.17. *Let C/K be a complete smooth curve of genus g over a field $K \supset k$. Let $\nabla_{/K} : L \to L \otimes \Omega^1_{C/K}(\mathcal{D})$ be a connection, such that*

$$\Big(L \to L \otimes \Omega^1_{C/K}(\mathcal{D})\Big) \to \Big(j_*L|(C-D) \to j_*L \otimes \Omega^1_{C/K}|(C-D)\Big)$$

is a quasiisomorphism. This implies that the divisor $\mathcal{D}$ is minimal such that $\nabla|_{C-D}$ extends with values in $\Omega^1_{C/K}(\mathcal{D})$ (see section 4, (4.2)). We also assume that L has degree 0, and that the connection on $L|_{C-D}$ lifts to an integrable, absolute (i.e. $/k$) connection $\tilde{\nabla}$. Then

$$\nabla|_{\mathcal{D}} : L|_{\mathcal{D}} \to L \otimes \Omega^1_{C/K}(\mathcal{D})|_{\mathcal{D}} \tag{3.60}$$

is an $\mathcal{O}_{\mathcal{D}}$-linear isomorphism and determines a trivialization of

$$\Omega^1_{C/K}(\mathcal{D})|_{\mathcal{D}}$$

Write $J_{\mathcal{D}}$ for the generalized jacobian and $J_{\mathcal{D},N}$ for the torseur of divisors of degree $N := 2g-2+\deg\mathcal{D}$. The above trivialization corresponds to a K-point $b \in J_{\mathcal{D},N}$. Write $\pi_N : (C-D)^N \to J_{\mathcal{D},N}$ for the natural map, and let $(L_N, \tilde{\nabla}_N)$ be the evident bundle and absolute connection on $(C-D)^N$. Then there exists a unique invariant, absolute connection $(\mathcal{L}, \Phi)$ on $J_{\mathcal{D},N}$ such that $\pi_N^*(\mathcal{L}, \Phi) = (L_N, \tilde{\nabla}_N)$. Moreover, we have

$$\Big(\det(H^*_{DR}(C-D,(L,\nabla)), \nabla_{GM}\Big)^{-1} \cong (\mathcal{L}, \Phi)|_{\{b\}} + \tau(L) \tag{3.61}$$

where $\tau(L)$ is as in definition 3.16.

Remark 3.18. *2-torsion also occurs in the determinant of de Rham cohomology for the trivial connection* [2]. *By virtue of the following lemma this can only happen when the variety has even dimension.*

Lemma 3.19. *Let X/K be a smooth projective variety of odd dimension $n = 2m+1$ over a function field in characteristic 0. Then the Gauß-Manin determinant*

$$\det(H_{DR}(X/K), d)$$

is trivial in $\Omega^1_K/d\log K^*$.

Proof. The strong Lefschetz theorem identifies the determinant connections on H^p_{DR} and H^{2n-p}_{DR} so we need only consider the connection on $\det H^n_{DR}$. As well known, the Poincaré duality morphism

$$\varphi : H^n_{DR}(X/K) \otimes H^n_{DR}(X/K) \to H^{2n}_{DR}(X/K) = K$$

is compatible with the Gauß-Manin connection, which is trivial on $H^{2n}_{DR}(X/K) = K$. On the other hand, it is alternating, thus its determinant

$$\det(\varphi) : \det(H^n_{DR}(X/K)) \otimes \det(H^n_{DR}(X/K)) \to K$$

fulfills

$$\det(\varphi)(e \otimes e) = p^2 \cdot 1$$

where $p \in K^*$ is the Pfaffian of the determinant of φ, written in the basis e. Thus if $\nabla(e) = \alpha \otimes e$, one has

$$\det(\varphi)(\nabla(e \otimes e)) = 2\alpha p^2 \cdot 1 = 2pd(p) \cdot 1.$$

Thus $\alpha = d \log p$ and the determinant of the Gauß-Manin connection is trivial. □

4. Product and Trace

In this section, we introduce a product which is reminiscent of Deligne's product explained in [5].

We keep the notations of sections 2 and 3 for C/K, (L, ∇), $j : U = C - D \to C$, $\mathcal{D} = \sum m_i c_i$, and $\mathcal{D}' = \mathcal{D} - D$. Further,

$$\nabla : L \to L \otimes \Omega^1_U \tag{4.1}$$

is an absolute connection with vertical curvature $\nabla^2(L) \subset L \otimes \Omega^2_K \otimes K(X)$. Let $\nabla_{/K} : \mathcal{L} \to \mathcal{L} \otimes \Omega^1_{C/K}(\mathcal{D})$ be an extension of $(L, \nabla_{/K})$ such that

$$\{\mathcal{L} \to \mathcal{L} \otimes \Omega^1_{C/K}(\mathcal{D})\} \to \{j_*L \to j_*L \otimes \Omega^1_{U/K}\} \tag{4.2}$$

is a quasiisomorphism. We assume $\nabla_{/K}$ has a pole at every $c_i \in D$. Note this implies that $\nabla_{/K}$ does not factor through $\Omega^1_{C/K}(\mathcal{D} - c_i)$ for any i. Indeed, by assumption, the complex

$$j_*L/\mathcal{L} \to (j_*L/\mathcal{L}) \otimes \Omega^1_{C/K}(\mathcal{D}) \tag{4.3}$$

is acyclic. Take e a local basis of $\mathcal{L}$ at c_i and z a local parameter, and suppose the connection can be written locally as $\nabla_{/K} e = a(z)dz/z^{m-1}e$ with $m = m_i$. Then $\nabla_{/K}(z^{-1}e) = (a(z)dz/z^m - z^{-2}dz)e$. The assumption that $\nabla_{/K}$ does have a pole at c_i implies that $m \geq 2$, so $z^{-1}e$ would represent a nontrivial element in H^0 of the complex (4.3), a contradiction.

By lemma 3.1 we know that the verticality condition implies that the absolute connection extends as

$$\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega^1_C\langle D\rangle(\mathcal{D}'). \tag{4.4}$$

From now on, we fix such a $(\mathcal{L}, \nabla)$.

As we have seen, the map $\mathcal{L}|_{\mathcal{D}} \to \mathcal{L} \otimes \Omega^1_{C/K}(\mathcal{D})|_{\mathcal{D}}$ is function linear. Since the connection does not factor through lower order poles, this gives a trivialization (denoted $\mathrm{triv}(\nabla)$) of $\Omega^1_{C/K}(\mathcal{D})|_{\mathcal{D}}$. We have

$$\begin{gathered}(c_1(\Omega^1_{C/K}(\mathcal{D})), \mathrm{triv}\ (\nabla)) \in \mathbb{H}^1(C, \mathcal{O}^*_C \to \mathcal{O}^*_{\mathcal{D}}) \\ (\mathcal{L}, \nabla) \in \mathbb{H}^1(C, \mathcal{O}^*_C \to \Omega^1_C\langle D\rangle(\mathcal{D}')).\end{gathered} \tag{4.5}$$

The aim of this section is to define a product

$$\begin{aligned}\cup : \mathbb{H}^1(C, \mathcal{O}^*_C \to \mathcal{O}^*_{\mathcal{D}}) \times \mathbb{H}^1(C, \mathcal{O}^*_C \to \Omega^1_C\langle D\rangle(\mathcal{D}')) \\ \to \mathbb{H}^2(C, \mathcal{K}_2 \to \Omega^2_C)\end{aligned} \tag{4.6}$$

Here $\mathcal{K}_2$ is the Milnor sheaf associated to K_2, and the map $\mathcal{K}_2 \to \Omega^2_C$ is the $d\log$ map $\{a, b\} \mapsto \frac{da}{a} \wedge \frac{db}{b}$. For a more detailed study of characteristic classes for connections defined in the hypercohomology of such complexes, the reader is referred to [7]. In addition, we will define a trace

$$\mathrm{Tr}\ : \mathbb{H}^2(C, \mathcal{K}_2 \to \Omega^2_C) \to \Omega^1_K/d\log K^* \tag{4.7}$$

We write

$$A \cdot B := \mathrm{Tr}(A \cup B) \tag{4.8}$$

so for example

$$\begin{aligned}(c_1(\Omega^1_{C/K}(\mathcal{D})), \mathrm{triv}\ (\nabla)) \cdot (\mathcal{L}, \nabla) := \\ \mathrm{Tr}\ ((c_1(\Omega^1_{C/K}(\mathcal{D})), \mathrm{triv}\ (\nabla)) \cup (\mathcal{L}, \nabla))\end{aligned} \tag{4.9}$$

Let $\mathcal{O}^*_{C,\mathcal{D}} = \mathrm{Ker}\ (\mathcal{O}^*_C \to \mathcal{O}^*_{\mathcal{D}})$. Then

Lemma 4.1. *$d\log \mathcal{O}^*_{C,\mathcal{D}} \wedge \Omega^1_C\langle D\rangle(\mathcal{D}') \subset \Omega^2_C$*

Proof. Since $\mathcal{O}^*_{C,\mathcal{D}} \subset 1 + \mathcal{I}_{\mathcal{D}}$, where $\mathcal{I}_{\mathcal{D}}$ is the ideal sheaf of $\mathcal{D}$,

$$d\log \mathcal{O}^*_{C,\mathcal{D}} \subset \mathcal{O}_C d\mathcal{I}_{\mathcal{D}} \subset \Omega^1_C(*D).$$

Also one has $d\mathcal{I}_{\mathcal{D}} \subset \mathcal{I}_{\mathcal{D}} \otimes_{\mathcal{O}_C} \Omega^1_C\langle D\rangle$. Thus

$$d\log \mathcal{O}^*_{C,\mathcal{D}} \wedge \Omega^1_C\langle D\rangle(\mathcal{D}') \subset \mathcal{I}_D \otimes_{\mathcal{O}_C} \Omega^2_C\langle D\rangle \subset \Omega^2_C.$$

∎

We define $\cup$ by

$$\begin{aligned}(4.10)\qquad & \mathcal{O}^*_{C,\mathcal{D}} \cup \mathcal{O}^*_C \to \mathcal{K}_2 \\ & (\lambda, c) \mapsto \{\lambda, c\} \\ & \mathcal{O}^*_{C,\mathcal{D}} \cup \Omega^1_C\langle D\rangle(\mathcal{D}') \to \Omega^2_C \\ & (\lambda, \omega) \mapsto d\log\lambda \wedge \omega.\end{aligned}$$

Concretely, we can write the product in terms of Cech cocyles. Here $\mathcal{C}^i$ refers to Cech cochains, δ is the Cech coboundary, and d is a boundary in the complex:

$$(\lambda_{ij}, \mu_i) \in (\mathcal{C}^1(\mathcal{O}^*_C) \times \mathcal{C}^0(\mathcal{O}^*_{\mathcal{D}}))_{d-\delta}$$
$$(c_{ij}, \omega_i) \in (\mathcal{C}^1(\mathcal{O}^*_C) \times \mathcal{C}^0(\Omega^1_C\langle D\rangle(\mathcal{D}'))_{d-\delta}$$

one has

$$\begin{aligned}(4.11)\qquad (\lambda, \mu) \cup (c, \omega) &= (\{\lambda_{ij}, c_{jk}\}, d\log\lambda_{ij} \wedge \omega_j, -d\log\tilde{\mu_i} \wedge \omega_i) \\ &\in (\mathcal{C}^2(\mathcal{K}_2) \times \mathcal{C}^1(\Omega^2_C\langle D\rangle(\mathcal{D}')) \times \mathcal{C}^0(\Omega^2_C\langle D\rangle(\mathcal{D}')/\Omega^2_C))_{d+\delta}\end{aligned}$$

where $\tilde{\mu_i} \in \mathcal{C}^0(\mathcal{O}^*_C)$ is a local lifting of μ_i. Note we have replaced the complex $\mathcal{K}_2 \to \Omega^2_C$ with the quasiisomorphic complex

$$\mathcal{K}_2 \to \Omega^2_C\langle D\rangle(\mathcal{D}') \to \Omega^2_C\langle D\rangle(\mathcal{D}')/\Omega^2_C.$$

Proposition 4.2. *The product* $\cup$ *extends to*

$$\begin{aligned}\cup : \mathbb{H}^1(C, \mathcal{O}^*_C \to \mathcal{O}^*_{\mathcal{D}}) \times \mathbb{H}^1(C, j_*\mathcal{O}^*_U \to \Omega^1_C\langle D\rangle(\mathcal{D}')) \\ \to \mathbb{H}^2(C, \mathcal{K}_2 \to \Omega^2_C).\end{aligned}$$

Proof. The map

$$\mathbb{H}^1(C, \mathcal{O}^*_C \to \Omega^1_C\langle D\rangle(\mathcal{D}')) \to \mathbb{H}^1(C, j_*\mathcal{O}^*_U \to \Omega^1_C\langle D\rangle(\mathcal{D}'))$$

is surjective, and its kernel is the $\mathbb{Z}$-module generated by $(\mathcal{O}(D_i), d_i)$ where d_i is the connection with logarithmic poles along D_i with residue -1. Let z_1 be a local coordinate around c_1. Let U_i be a Cech covering of C, with $c_1 \in U_1 \subset V_1$, and $c_1 \notin U_i, i \neq 1$. Assume c_1 is the only zero or pole of z_1 on U_1. Let

$$(\lambda, \mu) \in \mathbb{H}^1(C, \mathcal{O}^*_C \to \mathcal{O}^*_{\mathcal{D}})$$

be a Cech representative of a class in $\mathbb{H}^1(C, \mathcal{O}^*_C \to \mathcal{O}^*_{\mathcal{D}})$. Then (c_{ij}, ω_i) with $c_{1j} = z_1^{-1}$, $c_{ij} = 1$ for $i \neq 1$, $\omega_1 = -d\log z_1$, $\omega_i = 0$ for $i \neq 1$ is a Cech representative of $(\mathcal{O}(D_1), d_1)$. Thus considering $Z \in \mathcal{C}^0(\mathcal{O}_C[z_1^{-1}]^*)$ with $Z_1 = z_1$ and $Z_i = 1$ for $i \neq 1$, the cocyle of (4.7) is just the coboundary

$$(d-\delta)(\{\lambda_{ij}, Z_j\}, d\log\tilde{\mu_i} \wedge d\log Z_i) \in (d-\delta)(\mathcal{C}^1(\mathcal{K}_2) \times \mathcal{C}^0(\Omega^2_C\langle D\rangle(\mathcal{D}')).$$

(Note Z_j is invertible on U_{ij} for $i \neq j$ so the K_2-cochain is defined.) $\square$

Now we define the trace. We have (with standard K-theoretic notation, [1])

$$(4.12) \qquad H^2(C, \mathcal{K}_2) = 0$$

$$\text{Nm } : H^1(C, \mathcal{K}_2) = \{\oplus_{x \in C^{(1)}} \kappa(x)^*\}/\text{Tame}(K_2(K(C))) \to K^*$$

$$\sum_x \varphi_x \mapsto \Pi_x \text{Nm } (\varphi_x)$$

and of course $H^1(C, \Omega^2_C) = \Omega^1_K \otimes H^1(C, \Omega^1_{C/K}) = \Omega^1_K$. This defines

(4.13)

$$\text{Tr } : \mathbb{H}^2(C, \mathcal{K}_2 \to \Omega^2_C) = H^1(C, \Omega^2_C)/H^1(C, \mathcal{K}_2) \to \Omega^1_K/d\log K^*.$$

Lemma 4.3. *The trace*

$$\text{Tr } : \mathbb{H}^2(C, \mathcal{K}_2 \to \Omega^2_C) = \mathbb{H}^2(C, \mathcal{K}_2 \to \Omega^2_C\langle D\rangle(\mathcal{D}') \to \Omega^2_C\langle D\rangle(\mathcal{D}')/\Omega^2_C)$$
$$\to \Omega^1_K/d\log K^*$$

factors through

$$(4.14) \qquad \mathbb{H}^2\Big(C, \mathcal{K}_2 \to \Omega^1_K \otimes \Omega^1_{C/K}(\mathcal{D}) \to \Omega^1_K \otimes (\Omega^1_{C/K}(\mathcal{D})/\Omega^1_{C/K})\Big)$$
$$\cong \Omega^1_K/d\log K^*$$
$$\cong \Omega^1_K \otimes_K H^0(\mathcal{D}, \omega_{\mathcal{D}/K})/\mathbb{H}^1(C, \mathcal{K}_2 \to \Omega^1_K \otimes \Omega^1_{C/K}(\mathcal{D}))$$

where $\omega_{\mathcal{D}/K}$ is the relative dualizing sheaf of the scheme $\mathcal{D}$, containing $K \cong \omega_{D/K}$.

Proof. Note that

$$\Omega^1_K \otimes_K \Omega^1_{C/K}(\mathcal{D}) \cong \Omega^2_C\langle D\rangle(\mathcal{D}')/(\Omega^2_K \otimes \mathcal{O}_C(\mathcal{D}'))$$

so the complex in (4.14) is indeed a quotient. From the diagram

$$\begin{array}{ccccccccc} & & \mathcal{K}_2 & = & \mathcal{K}_2 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \Omega^1_K \otimes \Omega^1_{C/K} & \longrightarrow & \Omega^1_K \otimes \Omega^1_{C/K}(\mathcal{D}) & \longrightarrow & \Omega^1_K \otimes (\Omega^1_{C/K}(\mathcal{D})/\Omega^1_{C/K}) & \longrightarrow & 0 \end{array}$$

one deduces that the left hand side of (4.14) is isomorphic to

$$\mathbb{H}^2(C, \mathcal{K}_2 \to \Omega^1_K \otimes \Omega^1_{C/K}) \cong \text{coker}(H^1(C, \mathcal{K}_2) \to \Omega^1_K \otimes H^1(C, \Omega^1_{C/K})).$$

The right hand side here is identified under the norm with $\Omega^1_K/d\log K^*$, which proves the second equality. The third one comes from the map

$$\Omega^1_K \otimes \omega_{\mathcal{D}/K}[-2] \to \{\mathcal{K}_2 \to \Omega^1_K \otimes \Omega^1_{C/K}(\mathcal{D}) \to \Omega^1_K \otimes \omega_{\mathcal{D}/K}\}$$

and the vanishing of $\mathbb{H}^2(\mathcal{K}_2\Omega^1_K \otimes \Omega^1_C(\mathcal{D}))$. Note that this cumbersome way of writing this cohomology allows to write local contribution of a class in this cohomology group. $\square$

The first main result of this section is the following

Theorem 4.4. *Let $(\mathcal{L}, \nabla)$ and $(\mathcal{L}', \nabla')$ be two extensions of the vertical connection (L, ∇) on U as above satisfying the quasiisomorphism condition* (4.2). *Then, with notation as in* (4.10),

$$((c_1(\Omega^1_{C/K}), \text{triv } \nabla) \cdot (\mathcal{L}, \nabla)) = ((c_1(\Omega^1_{C/K}), \text{triv } \nabla') \cdot (\mathcal{L}', \nabla')).$$

Proof. The quasiisomorphism condition is local about each point of D, so we may assume our line bundles are $\mathcal{L}(\nu c) \subset \mathcal{L}$ for some $\nu < 0$ and $c \in D$.

Choose local coordinates z_i near c_i and a Cech covering U_i of C such that $c_i \in U_i$, $z_i \in \mathcal{O}^*(U_i - c_i)$, $c_i \notin U_j$ for $i \neq j$. Let us denote by

$$(c_1(\Omega^1_{C/K}), z_i) \in \mathbb{H}^1(C, \mathcal{O}^*_C \to \mathcal{O}^*_{\mathcal{D}})$$

the class defined by the local trivialization

$$\frac{dz_i}{z_i^{m_i}} : \mathcal{O}_{m_ic_i} \to \Omega^1_{C/K}(\mathcal{D}) \otimes \mathcal{O}_{m_ic_i}.$$

Let $(\mathcal{L}, \nabla) = (c_{ij}, \omega_i)$. Then $\omega_i = a_i \frac{dz_i}{z_i^{m_i}} + \frac{b_i}{z_i^{m_i-1}}$, with $a_i \in \mathcal{O}_C$ such that $a_i|_{m_ic_i} \in \mathcal{O}^*_{m_ic_i}$ and $b_i \in \Omega^1_K \otimes \mathcal{O}_C$. Suppose $c = c_i$. We drop the index i for convenience. One has

$$\begin{aligned} &((c_1(\Omega^1_{C/K}), \text{triv } \nabla) \cdot (\mathcal{L}, \nabla)) = \\ &(c_1(\Omega^1_{C/K}), z) \cdot (\mathcal{L}, \nabla) + \Big(0, 0, -d\log(a) \wedge (a\frac{dz}{z^m} + \frac{b}{z^{m-1}})\Big). \end{aligned} \tag{4.15}$$

where the last term is a cocycle as in (4.12) or the quotient complex (4.17). For $(\mathcal{L}(\nu c), \nabla(\nu c))$ one replaces a by $a - \nu z^{m-1}$, leaving b and m unchanged.

By proposition 4.2, one has

$$(c_1(\Omega^1_{C/K}), z) \cdot (\mathcal{L}, \nabla) = (c_1(\Omega^1_{C/K}), z) \cdot (\mathcal{L}(\nu c), \nabla(\nu c))$$

Thus

$$\begin{aligned} &((c_1(\Omega^1_{C/K}), \text{triv } \nabla) \cdot (\mathcal{L}, \nabla)) - ((c_1(\Omega^1_{C/K}), \text{triv } \nabla') \cdot (\mathcal{L}', \nabla')) \\ &= \Big(0, 0, (d(a - \nu z^{m-1}) - d(a)) \wedge \frac{dz}{z^m} + d\log(\frac{a - \nu z^{m-1}}{a}) \wedge \frac{b}{z^{m-1}}\Big) \\ &= \Big(0, 0, \frac{\nu(m-1) d\log z \wedge \frac{b}{a}}{(1 - \nu\frac{z^{m-1}}{a})}\Big). \end{aligned}$$

The nontrivial part in the last expression is computed in

$$H^0(\Omega^2_C\langle c\rangle/\Omega^2_C) \cong \Omega^1_c.$$

Computing using the residue at c we find the above difference is

$$\Big(0, 0, \nu(m-1)\frac{b(c)}{a(c)}\Big)$$

The verticality condition for the curvature reads

$$da \wedge \frac{dz}{z^m} + \frac{db}{z^{m-1}} - (m-1)\frac{dz}{z^m} \wedge b = 0 \in \Omega^1_K \otimes \Omega^1_{C/K}(\mathcal{D}).$$

In particular, $(da-(m-1)b)_{|c} = 0$. The difference of the two products is therefore $(0, 0, d\log a^\nu)$, which vanishes in $\Omega^1_K/d\log K^*$. □

Remark 4.5. *A version of the formula* (4.15) *in higher rank plays a central role in section 5.*

Suppose now

$$m_i = 1 \text{ for all } i. \tag{4.16}$$

In this case, the class $(c_1(\Omega^1_{C/K}(D), z_i) \in \mathbb{H}^1(C, \mathcal{O}^*_C \to \mathcal{O}^*_{\mathcal{D}})$ as defined in the proof of theorem 4.4 does not in fact depend on the choice of the local coordinate z_i. Indeed, the trivialization

$$\mathcal{O}_D \to (\Omega^1_{C/K}(D)/\Omega^1_{C/K} = \mathcal{O}_D), 1 \mapsto \frac{dz_i}{z_i}$$

is just the canonical identification given by the residue along c_i. In other words, the class $(c_1(\Omega^1_{C/K}(D), z_i)$ is what is denoted by $(c_1(\Omega^1_{C/K}(D), \mathrm{res}_D)$ in [2], and appears on the right hand side of the Riemann-Roch formula. The second main result of this section is

Theorem 4.6. *Let $(\mathcal{L}, \nabla)$ be as above, with $m_i = 1$ for all i. Then*

$$\det\Big(H^*_{DR}(U, L)),\ \textit{Gauß - Manin connection}\Big) = \\ - c_1\Big(\Omega^1_{C/K}(D), \mathrm{triv}\nabla\Big) \cdot (\mathcal{L}, \nabla).$$

Proof. Given the main result of [2], and lemma 3.19, the theorem is of course equivalent to

$$c_1(\Omega^1_{C/K}(D), \mathrm{res}_D) \cdot (\mathcal{L}, \nabla) = c_1(\Omega^1_{C/K}(D), \mathrm{triv}\ \nabla) \cdot (\mathcal{L}, \nabla). \tag{4.17}$$

Keeping the same notations as in the proof of theorem 4.4, one has

$$c_1(\Omega^1_{C/K}(D), \mathrm{res}_D) - c_1(\Omega^1_{C/K}(D), \mathrm{triv}\ \nabla) = (0, a_i)$$

and thus

$$(c_1(\Omega^1_{C/K}(D), \mathrm{res}_D) - c_1(\Omega^1_{C/K}(D), \mathrm{triv}\ \nabla)) \cdot (\mathcal{L}, \nabla) =$$
$$(0, 0, -d(a_i) \wedge d\log z_i - d\log a_i \wedge b_i) =$$
$$(0, 0, -d(a_i) \wedge d\log z_i).$$

This lies in $\Omega^1_K \otimes \omega_{D/K} = \Omega^1_K$ and by lemma 4.3, its trace factors through $\Omega^1_K \otimes H^1(C, \Omega^1_{C/K})$. But the image of $\gamma = \sum_i a_i d\log z_i \in \omega_{D/K}$ in $H^1(C, \Omega^1_{C/K})$ is the relative Atiyah class $\mathrm{at}_{/K}(\mathcal{L})$ ([6], appendix B), thus the image of $d\gamma = \sum_i d(a_i) \wedge d\log z_i \in \Omega^1_K \otimes \omega_{D/K}$ in $\Omega^1_K \otimes H^1(C, \Omega^1_{C/K})$ is $d(\mathrm{at}(\mathcal{L}))$, where $\mathrm{at}(\mathcal{L}) \in H^1(C, \Omega^1_C)$ is the absolute Atiyah class of $\mathcal{L}$. Indeed, $d : H^1(C, \Omega^1_C) \to \Omega^1_K \otimes H^1(C, \Omega^1_{C/K})$ factors through $H^1(C, \Omega^1_{C/K})$ by Hodge theory. On the other hand, if $c_{ij} \in \mathcal{C}^1(\mathcal{O}^*_C)$ is a cocyle for $\mathcal{L}$, then $d\log c_{ij} \in \mathcal{C}^1(\Omega^1_C)$ is a cocyle for $\mathrm{at}(\mathcal{L})$, and consequently, $d(\mathrm{at}(\mathcal{L})) = 0$. □

We want to explain briefly a fundamental compatibility satisfied by the pairing (4.8). Let $b = \sum b_i$ be a 0-cycle on C with support disjoint from $\mathcal{D}$, and let $\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega^1_C < D > (\mathcal{D}')$ be an absolute, integrable connection. We can interpret $\nabla|_{b_i} \in \Omega^1_{K(b_i)}/d\log K(b_i)^*$.

Proposition 4.7. *With notation as above, let $[b] \in \mathbb{H}^1(C, \mathcal{O}^*_C \to \mathcal{O}^*_{\mathcal{D}})$ be the class of the 0-cycle b. Then*

$$[b] \cdot (\mathcal{L}, \nabla) = \sum_i \mathrm{Tr}_{K(b_i)/K}(\nabla|_{b_i}) \in \Omega^1_K/d\log K^*.$$

Let $(\mathcal{L}_0, \nabla_0)$ be the invariant connection on $J_{\mathbb{D}}$ which pulls back to $(\mathcal{L}, \nabla)$ via the cycle map $i : (C - D) \to J_{\mathcal{D}}$ (proposition 2.17). Then

$$[b] \cdot (\mathcal{L}, \nabla) = \mathrm{Tr}_{i_0(b)/}(\mathcal{L}_0, \nabla_0)|_{i_0(b)}.$$

Proof. One reduces easily to the case b is a single K-point. Let U_2 be a Zariski-open set containing D and b. Shrink U_2 if necessary so there exists $z \in H^0(\mathcal{O}_{U_2})$ with $z|_{\mathcal{D}} = 1$ and $(z) = b$. Let $U_1 = C - \{b\} - \mathcal{D}$ so $C = U_1 \cup U_2$. Shrinking the U_i if necessary, we can assume $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$, so $(\mathcal{L}, \nabla)$ is represented by some cocycle $(\mu_{12}, \omega_1, \omega_2)$. Then

$$\nabla|_b = \omega_2|_b \in \Omega^1_K/d\log K^*$$

On the other hand, by the definition (4.12), $[b] \cdot (\mathcal{L}, \nabla)$ is represented by the image of the cocycle

$$d\log(z) \wedge \omega_2|_{U_{12}} \in H^1(C, \Omega^2_C) \to \Omega^1_K \otimes_K H^1(C, \Omega^1_{C/K}) \cong \Omega^1_K.$$

Write $\omega_2 = \omega_2(b) + z\eta_2$ with η_2 regular on U_2. Since $d\log(z) \wedge z\eta_2$ extends to U_2, it is homologous to zero, so

$$[b] \cdot (\mathcal{L}, \nabla) = \omega_2(b)[b] \in \Omega^1_K \otimes \mathbb{H}^1(C, \Omega^1_{C/K}) \mapsto \omega_2(b) \in \Omega^1_K.$$

Now, since one obviously has

$$\mathrm{Tr}(\nabla|_{b_i}) = \mathrm{Tr}(\nabla_0)|_{i_0(b_i)}$$

and the translation ∇_0 is invariant, the second equality is a direct interpretation of the first one. □

Now we can formulate and prove a variant of theorem 3.17.

Theorem 4.8. *Let* $(C/K, U/K, (L, \nabla), \mathcal{D})$ *be as in* (4.1), (4.2), (4.3). *Then*

(4.18)

$$\det\ (H^*_{DR}(U, L)) = -(c_1(\Omega^1_{C/K}(\mathcal{D}), \mathrm{triv}\ (\nabla)) \cdot (\mathcal{L}, \nabla) \in \Omega^1_K / d\log K^*$$

modulo torsion (see remark 3.18), where $\nabla_{/K} : \mathcal{L} \to \mathcal{L} \otimes \Omega^1_{C/K}(\mathcal{D})$ *is any extension of* $(L, \nabla_{/K})$ *having poles along all points of* $\mathcal{D}$ *such that*

$$\{\mathcal{L} \to \mathcal{L} \otimes \Omega^1_{C/K}(\mathcal{D})\} \to \{j_*L \to j_*(L \otimes \Omega^1_{U/K})\}$$

is a quasiisomorphism.

Proof. If all $m_i = 1$ this is just theorem 4.6. So we assume that $m_1 \geq 2$ in the sequel. Then as in the proof of theorem 4.4, replacing $\mathcal{L}$ by $\mathcal{L}(-c_1)$ changes a_1 to $(a_1 + z_1^{m_1-1})$ and keeps the rest unchanged. Thus the quotient complex

$$\mathcal{L}/\mathcal{L}(-c_1) \to \mathcal{L}/\mathcal{L}(-c_1) \otimes \Omega^1_{C/K}(\mathcal{D})$$

is $\mathcal{O}_{c_1}$-linear and the map is the multiplication by $a_1 \in \mathcal{O}^*_{c_1}$. In particular, $\mathcal{L}(\nu c_1)$ fulfills (4.2) for all $\nu \in \mathbb{Z}$ and taking $\nu = -\deg\ \mathcal{L}$, we may assume by theorem 4.4 that $\deg\ \mathcal{L} = 0$. If $H^0_{DR}(U, L) \neq 0$, then there is a meromorphic section φ of $\mathcal{L}$ verifying the flatness condition

$$d\varphi + \omega\varphi = 0.$$

This implies in particular that ω has at most logarithmic poles along D, which contradicts the condition $m_1 \geq 2$. On the other hand, $H^2_{DR}(U, L) = 0$ for dimension reasons, thus we can apply theorem 3.17 together with proposition 4.7 to obtain the result, after we have replaced $(\mathcal{L}, \nabla)$ by $(\mathcal{L}, \nabla) \otimes f^*\Big((\mathcal{L}, \nabla)|_{c_0}^{-1}\Big)$ to trivialize the connection at c_0 and applied the projection formula to this tensor connection. □

We finish this section with an example. Let $U = \mathbb{A}^1_K$ be the affine line over Spec$K$, with parameter t, and ∇ be a connection on the trivial bundle. Then up to a twist by a form of the base, ∇ has equation $A = df$, where $f = \sum_{i=1}^{m-1} a_i t^i$, $a_i \in K, a_{m-1} \neq 0$. Write $df = d_K f + f' dt$ with $d_k f = \sum_{i=1}^{m-2} da_i t^i$ and $f' = \sum_{i=1}^{m-2} i a_i t^{i-1}$. Let $b_i, i = 1, \ldots, (m-2)$ be the zeroes of f' (defined over K after some finite field extension), and let $N_\ell(\underline{b}) = \sum_{i=1}^{m-2} b_i^\ell$ be the Newton classes of the zeroes of f', which of course are expressable in the a_i already on Spec$K$. Then the main theorem says

$$\det(GM)^{-1} = \sum_{i=1}^{m-2} da_i N_i(\underline{b}).$$

5. A Formula in Higher Rank

In this section, we want to define a sort of non-commutative product of a higher rank connection with the Chern class of the dualizing sheaf of C with poles. The notations are as in the whole article: C is a curve defined over a function field K over an algebraically closed field k of characteristic 0, U is an open set such that $D = X - U = \sum_i c_i$ consists of $K-$rational points. Let $(\mathcal{E}, \nabla)$ be a rank r-connection on U with vertical curvature (2.58). Let m_i be the multiplicity of the relative connection at the point c_i, that is, the minimal multiplicity such ∇ factors

$$\nabla_{/K} : E \to E \otimes \Omega^1_{C/K}(\sum_i m_i c_i).$$

Lemma 3.1 no longer holds true in the higher rank case. We say that the poles of the global connection *behave well* if

$$\nabla : E \to E \otimes \Omega^1_C < D > (\mathcal{D}') \tag{5.1}$$

where $\mathcal{D} = \sum_i m_i c_i$ and $\mathcal{D}' = \mathcal{D} - D$.

Let $s = \{s_i\}$ be a trivialization of $\omega_{C/K}(\mathcal{D})|_{\mathcal{D}} \cong \omega_{\mathcal{D}/K}$. That is, $s_i \in \omega_{m_i c_i}$ and the map $1 \mapsto s_i$ is an isomorphism $\mathcal{O}_{\mathcal{D}} \cong \omega_{m_i c_i}$. For example, if z_i is a local parameter, one can take $s_i = \frac{dz_i}{z_i^{m_i}}$. We will abuse notation and write s_i also for a lifting of the trivialization to a local section of $\Omega^1_C < D > (\mathcal{D}')$. The local matrix of the connection has the shape

$$A_i = g_i s_i + \frac{\eta_i}{z_i^{m_i - 1}} \tag{5.2}$$

where g_i and η_i are $r \times r$ matrices with coefficients in $\mathcal{O}_C$ and $\Omega^1_K \otimes \mathcal{O}_C$ respectively. Note that the matrix of functions g_i depends only on the

lifting of s_i to a section of $\omega_{C/K}(\mathcal{D})$. (Indeed, the relative connection has matrix gs_i.)

We assume

$$\begin{aligned} &\mathrm{Image}(g_i) \in M(r \times r, \mathcal{O}_{\mathcal{D}}) \\ &\text{lies in } GL(r, \mathcal{O}_{\mathcal{D}}) \end{aligned} \tag{5.3}$$

Under this assumption, we define

$$\begin{aligned} &\{c_1(\omega(\mathcal{D})), \nabla\} := \\ c_1(\omega(\mathcal{D}), s_i) \cdot \det(\nabla) - &\sum_i \mathrm{res}\ \mathrm{Tr}(dg_i g_i^{-1} A_i) \\ &\in \Omega^1_K / d\log K^* \end{aligned} \tag{5.4}$$

Conjecture 5.1. *Assuming* (5.1) *and* (5.3)*, we have*

$$\det H^*_{DR}(U, (\mathcal{E}, \nabla))^{-1} = \{c_1(\omega(\mathcal{D})), \nabla\} \in (\Omega^1_K / d\log K^*) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We discuss the assumption (5.3) (see proposition 5.6) at the end of this section. The assumption (5.1) on the poles behaving well is not very well understood. It reflects a sort of stability in all possible directions for the poles of the global connection.

First, we justify the conjecture by establishing some rather surprising invariance properties for $\{c_1(\omega(\mathcal{D})), \nabla\}$.

Lemma 5.2. *Fix an index i and write the connection matrix locally in the form*

$$A = g\frac{dz}{z^m} + \frac{\eta}{z^{m-1}}$$

where g is an invertible matrix of functions and η is a matrix with entries from $\Omega^1_K \otimes \mathcal{O}_C$. Then

1. *res Tr*$(dgg^{-1}A) =$ *res Tr*$(dgg^{-1}\frac{\eta}{z^{m-1}})$.
2. $[\eta, g]z^{1-m}$ *has no pole at the point $z = 0$.*

Proof. The assumption that the curvature is vertical implies

$$dA = dg\frac{dz}{z^m} + d\Big(\frac{\eta}{z^{m-1}}\Big) \equiv A^2 = [\eta, g]\frac{dz}{z^{2m-1}} \mod \Omega^2_K \otimes \mathcal{O}_C[z^{-1}] \tag{5.5}$$

Multiplying through by z^m and contracting against $\frac{\partial}{\partial z}$ we deduce 2.

For 1, we must show res $\mathrm{Tr}(dg\frac{dz}{z^m}) = 0$. From 5.5, using $\mathrm{Tr}[g, \eta] = 0$ we reduce to showing res $\mathrm{Tr}d(\frac{\eta}{z^{m-1}}) = 0$. Since η has entries Ω^1_K, one has

$$\mathrm{res}\ \mathrm{Tr}d\Big(\frac{\eta}{z^{m-1}}\Big) = \mathrm{res}\ \mathrm{Tr}d_{C/K}\Big(\frac{\eta}{z^{m-1}}\Big).$$

And the residue of an exact form is vanishing. $\square$

Lemma 5.3. $\{c_1(\omega(\mathcal{D})), \nabla\}$ *is independent of the choice of the trivializations* s_i.

Proof. First we show independence of the choice of lifting of s. As remarked above, g is determined by the local lifting of s to $\omega(\mathcal{D})$, so $\{c_1(\omega(\mathcal{D})), \nabla\}$ depends only on that choice. If s and s' are two such local liftings, with $gs = g's'$, we have

$$dgg^{-1} = dg'g'^{-1} + d\log(\frac{s'}{s})I = dg'g'^{-1} + z^m h$$

for some $h \in M_r(\mathcal{O}_C)$. It follows immediately that

$$\text{res Tr}(dgg^{-1}A) = \text{res Tr}(dg'g'^{-1}A)$$

as desired.

Next we show independence of the choice of trivializations themselves. Let f be a rational function on C whose divisor (f) is disjoint from the singular locus of ∇. It will suffice to show that s and fs as trivializations give rise to the same invariant, i.e.

$$\begin{aligned}(5.6)\quad c_1(\omega(\mathcal{D}), s)\cdot\det(\nabla) - \sum_i \text{res Tr}(dg_i g_i^{-1}A_i) = \\ c_1(\omega(\mathcal{D}), fs)\cdot\det(\nabla) - \sum_i \text{res Tr}((dg_i g_i^{-1} - dff^{-1}I)A_i)\end{aligned}$$

Recall we can calculate $c_1(\omega(\mathcal{D}), s)\cdot\det(\nabla)$ by choosing δ a divisor in the linear series $\omega(\mathcal{D})$ compatible with the rigidification s and then restricting $\nabla|_\delta$ and taking the norm to Spec$\,(K)$. Associated to the trivialization fs we may take the divisor $\delta + (f)$. It follows that

$$c_1(\omega(\mathcal{D}), fs)\cdot\det(\nabla) - c_1(\omega(\mathcal{D}), s)\cdot\det(\nabla) = \text{Norm}\det\nabla|_{(f)}$$

(To get this relation, one could have taken the formula (4.15) as well). On the other hand, since the formula depends only on the local behavior of f near $\mathcal{D}$, by suitably choosing f, we may assume ∇ is defined by A in a neighborhood of (f) and that $f \equiv 1$ modulo some large power of the maximal ideal at the finite set of points where the connection is not given by A. We can interpret $\text{Tr}(dff^{-1}A) \in \Omega^1_K \otimes \omega_{k(C)}$, so the sum of the residues over all closed points of C will vanish in Ω^1_K. Thus

$$\sum_i \text{res Tr}(dff^{-1}A_i) = -\sum_{(f)} \text{res Tr}(dff^{-1}A) = \text{Tr}(A|_{(f)})$$

Since the connection matrix for the determinant connection is the trace of the connection matrix, the contributions to (5.6) cancel.

$\square$

Now consider what happens to the expression

$$\{c_1(\omega(\mathcal{D})), \nabla\} \tag{5.7}$$

under a change of coordinates given by a matrix h of functions. We have

$$A \mapsto A' := hAh^{-1} + dhh^{-1} = g'\frac{dz}{z^m} + \frac{\eta'}{z^{m-1}} \tag{5.8}$$

Note that h is regular, so dhh^{-1} does not contribute to the polar part of the connection, i.e.

$$g' = hgh^{-1} + z^m a; \quad \eta' = h\eta h^{-1} + z^{m-1}b \tag{5.9}$$

with a and b regular.

We compute

$$\begin{aligned} dg'g'^{-1} &= d(hgh^{-1})hg^{-1}h^{-1} + ez^m + fz^{m-1}dz = \\ &= dhh^{-1} + hdgg^{-1}h^{-1} - hgh^{-1}dhg^{-1}h^{-1} + ez^m + fz^{m-1}dz \end{aligned} \tag{5.10}$$

with e and f regular. Thus

$$\begin{aligned} \text{res Tr}(dg'g'^{-1}A') &= \text{res Tr}(dg'g'^{-1}hAh^{-1}) = \\ &\text{res Tr}(h^{-1}dhA) + \text{res Tr}(dgg^{-1}A) - \text{res Tr}(h^{-1}dhg^{-1}Ag). \end{aligned} \tag{5.11}$$

Note

$$A - g^{-1}Ag = z^{1-m}(\eta - g^{-1}\eta g) = z^{1-m}g^{-1}(g\eta - \eta g) \tag{5.12}$$

From lemma 5.2,2, this expression is regular. Plugging into (5.11), we conclude.

Lemma 5.4. $\{c_1(\omega(\mathcal{D})), \nabla\}$ *is independent of the choice of basis for the bundle E.*

Remarks 5.5.
1. *The definition of $\{c_1(\omega(\mathcal{D})), \nabla\}$ was inspired by the calculations in section 4 (cf. formula* (4.15)*) for a rank one connection. The formula*

$$\det H^*_{DR}(U, (\mathcal{E}, \nabla))^{-1} = \{c_1(\omega(\mathcal{D})), \nabla\} \tag{5.13}$$

when applied to the rank 1 case, gives back the main theorem of this article.

2. *When $m_i = 1$ for all c_i, that is when ∇ has regular singularities, then the argument from theorem 4.6 (slighlty modified in the higher rank case) gives*

$$\{c_1(\omega(D)), \nabla\} = c_1(\omega(D), \text{res}_D) \cdot \det(\nabla) \tag{5.14}$$

where, as in theorem 4.6, res_D *refers to the natural trivialization coming from the residue.*

3. *Finally, twisting* ∇ *by* $f^*\alpha$, *where* $\alpha \in \Omega^1_K$ *comes from the base, changes the right hand side of the formula by*

$$(2g-2+n-\sum_i m_i)r\alpha = -\chi(H^*_{DR}(U,(\mathcal{E},\nabla_{/K})))\alpha,$$

as it should. Here $r = \mathrm{rank} E$.

Proposition 5.6. *Let* $\mathcal{D} \subset C$ *be a divisor on a smooth curve, and let* $f \in \mathcal{O}_{C,\mathcal{D}}$ *be a local defining equation for* $\mathcal{D}$. *Let*

$$\nabla : E \to E \otimes \omega(\mathcal{D})$$

be a connection, and write

$$g = \nabla|_{\mathcal{D}} : E/E(-\mathcal{D}) \to (E(\mathcal{D})/E) \otimes \omega$$

Let $j : E - \mathcal{D} \hookrightarrow E$ *be the inclusion and consider the connection*

$$j_*j^*\nabla : j_*j^*E \to j_*j^*E \otimes \omega.$$

There is a natural map

$$\iota : \{E \to E \otimes \omega\} \to \{j_*j^*\nabla : j_*j^*E \to j_*j^*E \otimes \omega\}.$$

The map ι *is a quasi-isomorphism if and only if for any* $n \geq 0$ *the natural map*

$$g - n \cdot id \otimes \frac{df}{f} : E(n\mathcal{D})/E((n-1)\mathcal{D}) \to (E((n+1)\mathcal{D})/E(n\mathcal{D})) \otimes \omega$$

given by $f^{-n}e \mapsto f^{-n}g(e) - nf^{-n-1}e \otimes df$ *is a quasi-isomorphism.*

In particular, if, every point of $\mathcal{D}$ *has multiplicity* ≥ 2, *then* g *is an isomorphism if and only if* ι *is a quasi-isomorphism.*

Proof. The usual exact sequence reduces us to showing the condition is equivalent to

$$j_*j^*E/E \xrightarrow{\cong} (j_*j^*E/E) \otimes \omega(\mathcal{D}).$$

Writing as usual $\mathcal{D} = \mathcal{D}' + D$ where D is the reduced divisor with support equal to the support of $\mathcal{D}$, we have a commutative square

$$\begin{tikzcd}
E/E(-\mathcal{D}) \arrow[r, "g - nid\otimes\frac{df}{f}"] \arrow[d, "\text{``}\cdot f^{-n}\text{''}", "\cong"'] & (E(\mathcal{D}')/E(-D)) \otimes \omega(D) \arrow[d, "\text{``}\cdot f^{-n}\text{''}", "\cong"'] \\
E(n\mathcal{D})/E((n-1)\mathcal{D}) \arrow[r, "g"] & (E(n\mathcal{D}+\mathcal{D}')/E((n-1)\mathcal{D}+\mathcal{D}'))\otimes\omega(D)
\end{tikzcd}$$

The map ι is a quasi-isomorphism if and only if for all $n \geq 0$ the map g on the bottom line of the above square is an isomorphism, and this will hold only if the top line is.

In particular, since $\frac{df}{f}$ has poles of order 1, if all multiplicities are ≥ 2, then ι is a quasiisomorphism if and only if g is an isomorphism. $\square$

We close those remarks by a numerical computation for $E = \oplus_1^r \mathcal{O}$ on $U = \mathbb{A}^1_K$, with parameter t . There is only one singular point at ∞. Let us write

$$A = \sum_{i=0}^{m-1} B_i t^i + \sum_{i=0}^{m-2} C_i t^i dt$$

where the B_i and the C_i are matrices with coefficients in Ω^1_K respectively K. The assumption 5.3 means that $C_{m-2} \in GL(r, K)$. We consider the cases $m = 2, 3$. For $m = 2$ both sides of the formula are 0, and for $m = 3$ they are equal to

$$\mathrm{Tr}(B_0 - B_1 C_0 C_1^{-1} + B_2 C_0 C_1^{-1} C_0 C_1^{-1}) \tag{5.15}$$

Note that $-C_0 C_1^{-1}$ is the zero of the "polynomial" $C_0 + C_1 t$, where t is a "variable" of matrices, and thus the formula could also be written as

$$\mathrm{Tr} A|_{\mathrm{zero}(C_0 + C_1 t)} \tag{5.16}$$

if it made sense. For higher m, the right hand side should be a sort of restriction of ∇ to the "Newton" classes of $\sum_{i=0}^{m-2} C_i t^i$.

In the remaining part of this section, we show that the product $\{c_1(\omega(\mathcal{D})), \nabla\}$ is a particular case of a more general product between higher rank connections and a larger class of trivializations along $\mathcal{D}$.

We consider the tuples $\{E, L, \nabla, \mathcal{D}, g\}$, where

$$\nabla : E \to E \otimes \Omega^1_C < D > (\mathcal{D}') \tag{5.17}$$

is a connection on a rank r vector bundle E. We denote by $\nabla|_{\mathcal{D}}$ the $\mathcal{O}_C$-linear map

$$\nabla : E \to E \otimes \Big(\Omega^1_C < D > (\mathcal{D}')/\Omega^1_C\Big). \tag{5.18}$$

Further, L is a rank 1 bundle, g is a trivialization

$$g : E|_{\mathcal{D}} \cong E \otimes L|_{\mathcal{D}}, \tag{5.19}$$

which fulfills

$$0 = [g, \nabla|_{\mathcal{D}}] : E|_{\mathcal{D}} \to E \otimes L|_{\mathcal{D}} \tag{5.20}$$

By lemma 5.2, if $L = \omega_C(\mathcal{D})$ and g is the trivialization $E|_{\mathcal{D}} \to E \otimes \Big(\omega_C(\mathcal{D})/\omega_C\Big)$ arising from the principal part of $\nabla_{C/K}$, then the condition 5.20 is fulfilled.

Let us introduce the cocyles of the tuple $\{E, L, \nabla, \mathcal{D}, g\}$, as in section 4. If E has cocycle $c_{ij} \in GL(r, \mathcal{O}_C)$, L has cocycle $\lambda_{ij} \in \mathcal{O}_C^*$, g has

cocyle $\mu_i \in GL(r, \mathcal{O}_{\mathcal{D}})$, and ∇ has cocyle $\omega_i \in M(r\times r, \Omega^1_C < D > (\mathcal{D}'))$ then one has

$$dc_{ij}c_{ij}^{-1} = \omega_i - c_{ij}\omega_j c_{ij}^{-1} \tag{5.21}$$

$$\mu_i = c_{ij}\mu_j c_{ij}^{-1}\lambda_{ij}. \tag{5.22}$$

The commutativity condition 5.20 then reads

$$[\mu_i, \omega_i] = 0. \tag{5.23}$$

Let $\tilde{\mu}_i \in GL(r, \mathcal{O}_C)$ be a local lifting ot $\mu_i \in GL(r, \mathcal{O}_{\mathcal{D}})$. Then one has

Theorem 5.7. *The cochain*

$$\Big\{\{\lambda_{ij}, \det(c_{jk})\}, d\log\lambda_{ij} \wedge \mathrm{Tr}(\omega_j), \mathrm{Tr}(-d\tilde{\mu}_i\tilde{\mu}_i^{-1} \wedge \omega_i)\Big\} \tag{5.24}$$

is a cocyle in

$$\mathcal{K}_2 \to \Omega^2_X\langle D\rangle(\mathcal{D}') \to \Big(\Omega^2_X\langle D\rangle(\mathcal{D}')/\Omega^2_X\Big) \tag{5.25}$$

and defines a cohomology class $\{c_1(L), g, \nabla\}$ *in*

$$\mathbb{H}^2\Big(C, \mathcal{K}_2 \to \Omega^2_X\langle D\rangle(\mathcal{D}') \to \Big(\Omega^2_X\langle D\rangle(\mathcal{D}')/\Omega^2_X\Big)\Big) \tag{5.26}$$

Proof. First

$$\begin{aligned}(5.27)\quad & \delta(d\log(\lambda) \wedge \mathrm{Tr}(\omega))(ijk) = \\ & d\log(\lambda_{ij}) \wedge \mathrm{Tr}(\omega_j) + d\log(\lambda_{jk}) \wedge \mathrm{Tr}(\omega_k) - d\log(\lambda_{ik}) \wedge \mathrm{Tr}(\omega_k) = \\ & d\log(\lambda_{ik}) \wedge \mathrm{Tr}(\omega_j - \omega_k) - d\log(\lambda_{jk}) \wedge \mathrm{Tr}(-\omega_k + \omega_j) = \\ & d\log(\lambda_{ij}) \wedge \mathrm{Tr}(\omega_j - \omega_k) = d\log(\lambda_{ij}) \wedge d\log(\det(c_{jk})).\end{aligned}$$

Next computing mod the ideal of $\mathcal{D}$ and so ignoring the tilde,

$$\begin{aligned}(5.28)\quad & \delta(\mathrm{Tr}(-d\tilde{\mu}_i\tilde{\mu}_i^{-1} \wedge \omega_i))(ij) = \mathrm{Tr}(-d\tilde{\mu}_j\tilde{\mu}_j^{-1} \wedge \omega_j + d\tilde{\mu}_i\tilde{\mu}_i^{-1} \wedge \omega_i) = \\ & \mathrm{Tr}\Big((d\lambda_{ij}\lambda_{ij}^{-1} \cdot I + c_{ij}^{-1}dc_{ij} - c_{ij}^{-1}d\mu_i\mu_i^{-1}c_{ij} - c_{ij}^{-1}\mu_i dc_{ij}c_{ij}^{-1}\mu_i^{-1}c_{ij}) \wedge \omega_j + \\ & + d\mu_i\mu_i^{-1} \wedge \omega_i\Big) = \mathrm{Tr}\Big((d\lambda_{ij}\lambda_{ij}^{-1} \cdot I + c_{ij}^{-1}dc_{ij} - c_{ij}^{-1}d\mu_i\mu_i^{-1}c_{ij} \\ & - c_{ij}^{-1}\mu_i dc_{ij}c_{ij}^{-1}\mu_i^{-1}c_{ij}) \wedge (c_{ij}^{-1}\omega_i c_{ij} - c_{ij}^{-1}dc_{ij}) + d\mu_i\mu_i^{-1} \wedge \omega_i\Big)\end{aligned}$$

By our commutation assumption 5.20

$$\mathrm{Tr}(c_{ij}^{-1}dc_{ij} \wedge c_{ij}^{-1}\omega_i c_{ij}) = \mathrm{Tr}(c_{ij}^{-1}\mu_i dc_{ij}c_{ij}^{-1}\mu_i^{-1}c_{ij} \wedge c_{ij}^{-1}\omega_i c_{ij}) \tag{5.29}$$

Also terms with no poles (i.e. terms not involving ω) can be ignored. We get

(5.30)
$$\delta(\mathrm{Tr}(-d\tilde{\mu}_i\tilde{\mu}_i^{-1}\wedge\omega_i))(ij) = d\lambda_{ij}\lambda_{ij}^{-1}\wedge\mathrm{Tr}(\omega_i) = d\lambda_{ij}\lambda_{ij}^{-1}\wedge\mathrm{Tr}(\omega_j)$$

which is what we want. Note the right hand equality holds because we are computing mod forms regular along $\mathcal{D}$, and $\mathrm{Tr}(\omega_i)\equiv\mathrm{Tr}(\omega_j)$ mod regular forms by the cocycle condition 5.21.

It remains to show that our cocyle 5.24 does not depend on the choice of the cocycles $\{c,\omega,\lambda,\mu\}$.

If λ_{ij} is replaced by $\lambda_{ij}\delta(\nu)_{ij}$, then the new 5.24 differs from the old one by the cochain

$$\delta\Big(\{\nu_i,\det(c_{ij})\}, d\log\nu_i\wedge\mathrm{Tr}(\omega_i)\Big).$$

If c_{ij} is replaced by $\gamma_i c_{ij}\gamma_j^{-1}$, then ω_i is replaced by $d\gamma_i\gamma_i^{-1}+\gamma_i\omega_i\gamma_i^{-1}$, and μ_i is replaced by $\gamma_i\mu_i\gamma_i^{-1}$. The commutativity relation implies $d\gamma_i\mu_i\gamma_i^{-1}-\gamma_i\mu_i\gamma_i^{-1}d\gamma_i\gamma_i^{-1}=0$. Then the new 5.24 differs from the old one by

$$\{\{\lambda_{ij},\det(\delta(\gamma))_{jk}\}, d\lambda_{ij}\wedge\mathrm{Tr}d\log(\gamma_i),$$

(5.31)
$$\mathrm{res}\ \mathrm{Tr}\Big(\gamma_i^{-1}d\gamma_i\omega_i d\mu_i\mu_i^{-1}\gamma_i^{-1}d\gamma_i-\mu_i\gamma_i^{-1}d\gamma_i\mu_i^{-1}\gamma_i^{-1}d\gamma_i-\mu_i\gamma_i^{-1}d\gamma_i\mu_i^{-1}\omega_i\Big)\}$$

Using the commutativity relation 5.20 as well, we see that this is this expression is the cochain

$$\delta\Big(\{\lambda_{ij},\det\gamma_j\},0,0\Big). \tag{5.32}$$

□

Now we consider the image under the map $f: C\to\mathrm{Spec}\,(K)$:

$$f_*(\{c_1(L),g,\nabla\})\in\Omega^1_K/d\log K^* \tag{5.33}$$

of $\{c_1(L),g,\nabla\}$. We want to study closedness for forms in the image of this map.

Lemma 5.8. *Let $\{a,b,c\}$ be a cocycle as in (5.24) representing a class in $\mathbb{H}^2(C,\mathcal{K}_2\to\Omega^2_C)$, with $db=0$. Then $df_*\{a,b,c\}=\mathrm{res}_{\mathcal{D}}(dc)\in\Omega^2_K$.*

Proof. Another representative of $\{a,b,c\}$ in

$$\mathbb{H}^1(C,\Omega^2_C<D>(\mathcal{D}')\to\Omega^2_C<D>(\mathcal{D}')/\Omega^2_C)/d\log(H^1(C,\mathcal{K}_2))$$

is of the shape $\{0,b+d\log\alpha,c\}$, thus its derivative in

$$\mathbb{H}^1(C,\Omega^3_C<D>(\mathcal{D}')\to\Omega^3_C<D>(\mathcal{D}')/\Omega^3_C)$$

is of the shape $\{0,0,dc\}$ since $db=0$. Then one applies the commutativity of the diagramm

$$\begin{array}{ccc} \mathbb{H}^1(C,\Omega^2_C\langle D\rangle(\mathcal{D}')\to\Omega^2_C\langle D\rangle(\mathcal{D}')/\Omega^2_C) & \xrightarrow{d} & \mathbb{H}^1(C,\Omega^3_C\langle D\rangle(\mathcal{D}')\to\Omega^3_C\langle D\rangle(\mathcal{D}')/\Omega^3_C) \\ \downarrow & & \downarrow \\ \Omega^1_K & \xrightarrow{d} & \Omega^2_K \end{array} \tag{5.34}$$

□

Theorem 5.9. *If $\nabla^2=0$, that is if ∇ is flat, then $f_*(\{c_1(L),g,\nabla\})\in\Omega^1_{K,\mathrm{clsd}}/d\log K^*$, that is the image is flat as well. In particular, this is true for $\{c_1(\omega(\mathcal{D})),\nabla\}$, as predicted by conjecture 5.1.*

Proof. Since, under the integrabiltiy assumption, one has in particular $d(d\log\lambda_{ij}\wedge\mathrm{Tr}\omega_j)=0$, one can apply lemma 5.8. One has to compute

$$\begin{aligned}\gamma &= -\mathrm{res}\ \mathrm{Tr}\ d(d\mu\mu^{-1}\omega)\\ &= -\mathrm{res}\ \mathrm{Tr}(d\mu\mu^{-1})^2\omega+\mathrm{res}\ \mathrm{Tr}\ d\mu\mu^{-1}d\omega\end{aligned} \tag{5.35}$$

We omit the indices since we compute only with one index. Note in the calculations which follow μ is regular and ω has poles along $\mathcal{D}$. We write $a\equiv b$ to indicate that the polar parts of a and b coincide.

The condition 5.20 implies

$$d\mu\omega+\mu d\omega\equiv d\omega\mu-\omega d\mu \tag{5.36}$$

thus

$$d\mu\mu^{-1}\omega+\mu d\omega\mu^{-1}\equiv d\omega-\omega d\mu\mu^{-1} \tag{5.37}$$

which implies

$$(d\mu\mu^{-1})^2\omega+d\mu d\omega\mu^{-1}\equiv d\mu\mu^{-1}d\omega-d\mu\mu^{-1}\omega d\mu\mu^{-1} \tag{5.38}$$

So taking the trace, one obtains

$$2\mathrm{Tr}((d\mu\mu^{-1})^2\omega)\equiv\mathrm{Tr}((d\mu\mu^{-1}-\mu^{-1}d\mu)d\omega) \tag{5.39}$$

Now, under the integrability assumption

$$d\omega=\omega\omega \tag{5.40}$$

(using (5.23)) one obtains

$$2\mathrm{Tr}((d\mu\mu^{-1})^2\omega)=0 \tag{5.41}$$

Now we consider the other term

$$\gamma'=\mathrm{Tr}(d\mu\mu^{-1}(\omega)^2) \tag{5.42}$$

Choosing a local basis of the bundle E, we write μ as a matrix M, and ω as matrix Ω. Then the condition 5.20 reads

$$M\Omega = \Omega M \tag{5.43}$$

and one has

$$\gamma' = \mathrm{Tr}(dMM^{-1}\Omega^2) \tag{5.44}$$

The condition 5.43 implies

$$\Omega^2 = M^{-1}\Omega^2 M \tag{5.45}$$

thus

$$\gamma' = \mathrm{Tr}(dM\Omega^2 M^{-1}) = \mathrm{Tr}(M^{-1}dM\Omega^2) \tag{5.46}$$

On the other hand, differentiating the condition 5.43, one obtains

$$\begin{gathered} M^{-1}dM\Omega^2 + d\Omega\Omega = \\ M^{-1}d\Omega M\Omega - M^{-1}\Omega dM\Omega = \\ M^{-1}d\Omega\Omega M - \Omega M^{-1}dM\Omega \end{gathered} \tag{5.47}$$

So taking the trace, one obtains

$$\gamma' = \mathrm{Tr}(M^{-1}dM\Omega^2) = -\mathrm{Tr}(M^{-1}dM\Omega^2) = -\gamma' \tag{5.48}$$

Thus $\gamma' = 0$.

□

6. Rank 1 Irregular Connections in Arbitrary Dimension on Projective Manifolds

Let X be a smooth, projective variety in characteristic 0, and let $D \hookrightarrow X$ be a normal crossings divisor. Given $m_i \geq 0$, define $C(X, D, \underline{m})$ to be the group of isomorphism classes of line bundles $\mathcal{L}$ on X together with an integrable connection

$$\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega^1_X < D > (\mathcal{D})$$

where $\mathcal{D} = \sum_i m_i D_i$. Define for $m_i \geq 1$

$$\mathrm{Irreg}(X, D, \underline{m}) := C(X, D, \underline{m})/C(X, D, \underline{0}).$$

Theorem 6.1. *With notation as above, there is a canonical isomorphism*

$$\mathit{Irreg}(X, D, \underline{m}) \cong \Gamma(X, \mathcal{O}_X(\mathcal{D})/\mathcal{O}_X),$$

where $\mathcal{O}_X(\mathcal{D})/\mathcal{O}_X$ is the sheaf of principal parts of degree $\leq m_i$ along D_i.

Lemma 6.2. *Exterior differentiation induces an isomorphism*

$$d : \mathcal{O}_X(\mathcal{D})/\mathcal{O}_X \overset{\cong}{\to} \Omega^1_X < D > (\mathcal{D})_{\rm clsd}/\Omega^1_X < D >_{\rm clsd}$$

proof of lemma. We first check injectivity. Let $x = 0$ be a local defining equation for D. Suppose for some n with $1 \le n \le m$ we had

$$d(\frac{a}{x^n}) = \frac{1}{x^n}(da - na\frac{dx}{x}) \in \Omega^1_X < D >$$

Multiplying by x^n and taking residue along D, it would follow that $a|_D = 0$, i.e. $\frac{a}{x^n} = \frac{b}{x^{n-1}}$.

To show surjectivity, write a local section of $\Omega^1 < D > (nD)_{\rm clsd}$ (here $1 \le n \le m$) in the form

$$\omega = \frac{adx}{x^{n+1}} + \frac{B}{x^n} \tag{6.1}$$

where B does not involve dx. Replacing ω by $\omega + d(\frac{a}{nx^n})$, we can assume

$$\omega = \frac{adx + B}{x^n}.$$

Then

$$0 = d\omega = \frac{da \wedge dx + dB - n\frac{dx}{x} \wedge B}{x^n}.$$

Multiplying by x^n and taking residue along D, we see $B|_D = 0$. Since B does not involve dx, it follows that $B = xC$ and ω can be written

$$\omega = \frac{adx}{x^n} + \frac{C}{x^{n-1}}$$

Comparing with (6.1), we have lowered the order of pole by 1. This process continues until ω has log poles. $\square$

proof of theorem 6.1. Using the lemma, we get a diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_X^* & = & \mathcal{O}_X^* & \to & 0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Omega^1_X < D >_{\rm clsd} & \to & \Omega^1_X < D > (\mathcal{D})_{\rm clsd} & \to & \mathcal{O}_X(\mathcal{D})/\mathcal{O}_X & \to & 0 \end{array}$$

We view this as a diagram of complexes written vertically. Using the standard hypercohomological interpretation of line bundles with connection, this yields an exact sequence

$$0 \to C(X, D, 0) \to C(X, D, \underline{m}) \to H^0(X, \mathcal{O}_X(\mathcal{D})/\mathcal{O}_X)$$
$$\xrightarrow{\delta} \mathbb{H}^2(X, \mathcal{O}_X^* \to \Omega^1_X < D >_{\rm clsd})$$

We claim the map δ above is zero. In the derived category, δ factors

$$\mathcal{O}_X(\mathcal{D})/\mathcal{O}_X \xrightarrow{\cong} \frac{\Omega^1_X < D > (\mathcal{D})_{\mathrm{clsd}}}{\Omega^1_X < D >_{\mathrm{clsd}}} \xrightarrow{\partial} \Omega^1_X(< D >)_{\mathrm{clsd}}[1]$$
$$\to \{\mathcal{O}^*_X \to \Omega^1_X(< D >)_{\mathrm{clsd}}\}[2].$$

We have a factorization of ∂:

$$H^0(\mathcal{O}_X(\mathcal{D})/\mathcal{O}_X) \to H^1(\mathcal{O}_X) \xrightarrow{\tilde{d}} H^1(\Omega^1_{X,\mathrm{clsd}}) \to H^1(\Omega^1_X < D >_{\mathrm{clsd}}),$$

so it suffices to show the map $\tilde{d}$ is zero. By Hodge theory, the composition

$$H^1(\mathcal{O}_X) \xrightarrow{\tilde{d}} H^1(\Omega^1_{X,\mathrm{clsd}}) \xrightarrow{\iota} \mathbb{H}^1(\Omega^1_X \to \Omega^2_X)$$

is zero, and the map ι is injective as the complex $\{\Omega^1_X/\Omega^1_{X,\mathrm{clsd}} \to \Omega^2_X\}$ is quasi-isomorphic to the complex $\{0 \to \Omega^2_X/\Omega^2_{X,\mathrm{exact}}\}$, and in particular starts in degree 1. $\square$

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